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具有饱和传染率的脉冲免疫接种 SIRS模型分析

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摘要: 为制定合理的免疫接种策略,有效地防止传染病的产生和蔓延,研究了具有饱和传染率的脉冲免疫接种 SIRS模型的动力学行为. 利用 Floquet乘子理论和脉冲微分方程比较得到无病周期解的存在性和全局渐近稳定性;利用分支定理得到正周期解存在的分支参数. 结果表明,对于所研究的系统,只有当免疫接种率 $\theta > \theta^*$,或者脉冲免疫周期 $\tau < \tau^*$ 时,疾病消除;而当 $\tau > \tau_0$ 时,疾病会周期性地发生,形成地方病.

关键词: 脉冲;免疫接种; SIRS模型; 全局渐近稳定性; 周期解
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0 引 言

关于传染病模型的研究目前已取得大量成果,但是,所涉及的模型大多数是常微分方程或时滞微分方程. 通常情况下,为了有效地控制传染病的发生和流行,往往需要在给定的时间点进行免疫接种,因此,一些传染病模型用具有脉冲的微分方程描述更符合实际. 目前,所见的具有脉冲免疫接种的传染病模型的研究工作主要有 Roberts等和 Stone等的研究. Roberts等研究了出生具有脉冲的 SI传染病模型,给出了无病周期解的存在性和局部稳定性[1]; Stone等研究了具有脉冲预防接种 SIR模型,得到了基本再生数,并证明了无病周期解的局部渐近稳定性[2]. 此外,文献[3]研究了具有标准传染率的脉冲预防接种 SIRS传染病模型无病周期解的全局渐近稳定性. 总而言之,具有脉冲免疫接种的传染病模型的研究成果不多,也不完整,因此,本文研究具有饱和传染率的脉冲免疫接种 SIRS模型,得到无病周期解全局渐近稳定的充分条件,并应用分支理论研究正周期解的存在性,得到周期解存在的分支参数.

1 模型的建立

具有饱和传染率的脉冲免疫接种 SIRS模型流程图见图 1.

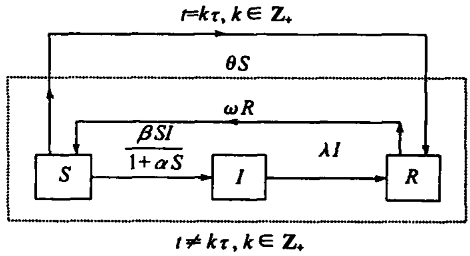

图 1 具有脉冲免疫接种的 SIRS模型流程图
Fig. 1 The flow chart for the SIRS model with pulse vaccination

相应的传染病动力模型为

$$\left\{\begin{array}{l}\left.\begin{array}{l}\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = -\dfrac{\beta}{1+\alpha S(t)}S(t)I(t) + \omega R(t)\\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \dfrac{\beta}{1+\alpha S(t)}S(t)I(t) - \lambda I(t)\\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \lambda I(t) - \omega R(t)\end{array}\right\} t = k\tau \\ \left.\begin{array}{l}S(t^+) = (1-\theta)S(t)\\ I(t^+) = I(t)\\ R(t^+) = R(t) + \theta S(t)\end{array}\right\} t \neq k\tau\end{array}\right. \tag{1}$$

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这里把总人口 $N(t)$ 分为易感者 $S(t)$、染病者 $I(t)$ 和恢复者 $R(t)$ 三类.系统(1)的前 3个方程相加得 $\mathrm{d}N(t)/\mathrm{d}t=0$,所以 $N(t)$ 是一个常数.为研究方便,不妨令 $N(t)=S(t)+I(t)+R(t)=1$. $\frac{U}{1+TS(t)}$ 代表饱和传染率系数,λ 代表移除率系数,k代表失去免疫率系数,θ 是免疫接种率,f是脉冲免疫接种周期,$k\in \mathbf{Z}_+$,T、U、k、λ、θ 均为正常数.系统(1)的可行区域为 $K=\{(S(t),I(t),R(t))\in \mathbf{R}^3 | 0\leqslant S(t),I(t),R(t)\leqslant 1$, 且 $S(t)+I(t)+R(t)=1\}$.

为研究方便,可考虑系统(1)的等价系统

$$\begin{cases}\left.\begin{aligned}\frac{\mathrm{d}S(t)}{\mathrm{d}t}&=-\frac{U}{1+TS(t)}S(t)I(t)+\\&\quad k(1-S(t)-I(t))\\\frac{\mathrm{d}I(t)}{\mathrm{d}t}&=\frac{U}{1+TS(t)}S(t)I(t)-\lambda I(t)\end{aligned}\right\}\ t=kf\\\left.\begin{aligned}S(t^+)&=(1-\theta)S(t)\\I(t^+)&=I(t)\end{aligned}\right\}\ t\neq kf\end{cases}\tag{2}$$

该系统的可行区域为 $D=\{(S(t),I(t))\in \mathbf{R}^2 | 0\leqslant S(t),I(t)\leqslant 1$,且 $S(t)+I(t)\leqslant 1\}$.

2　无病周期解

2.1　无病周期解的存在性

当 $I(t)=0$时,系统(2)变为

$$\begin{cases}\frac{\mathrm{d}S(t)}{\mathrm{d}t}=k(1-S(t));t\neq kf\\S(t^+)=(1-\theta)S(t);t=kf\end{cases}\tag{3}$$

系统(3)在区间 $kf\leqslant t\leqslant (k+1)f$上的解为

$$S(t)=\begin{cases}1-(1-S(kf))\exp(-k(t-kf));\\\qquad kf\leqslant t<(k+1)f\\S((k+1)f^+)=(1-\theta)S((k+1)f);\\\qquad t=(k+1)f\end{cases}$$

设 $F: S(kf)\rightarrow S((k+1)f)$ 是一个映射,满足

$$S((k+1)f^+)=F(S(kf))=(1-\theta)[1-(1-S(kf))\exp(-kf)]$$

该映射有惟一不动点

$$S^*=F(S^*)=\frac{(1-\theta)(1-\exp(-kf))}{1-(1-\theta)\exp(-kf)}\tag{4}$$

不动点 S^* 是易感者 $S(t)$ 在 $t=(k+1)f$处以 f 为周期的循环点.

$S^*>S>0$时,有 $S^*>F(S)>S$; $S>S^*$ 时,有 $S>F(S)>S^*$,所以不动点 S^* 全局渐近稳定,从而由脉冲免疫接种导出的数列 $\{S(kf)\}$ 必收敛于 S^*.因此,系统(2)在 $kf\leqslant t<(k+1)f$ 内有无病 f-周期解 $(\tilde{S}(t),0)$,其中

$$\tilde{S}(t)=1-\frac{\theta\exp(-k(t-kf))}{1-(1-\theta)\exp(-kf)}\tag{5}$$

2.2　无病周期解的稳定性

定理 1　当 $R'<1$时,系统(2)的无病 f-周期解 $(\tilde{S}(t),0)$ 是局部渐近稳定的,其中 $R'=\frac{U(\exp(kf)-1)}{\lambda((1+T)(\exp(kf)-1)+\theta)}$.

证明　作变换 $x(t)=S(t)-\tilde{S}(t)$, $y(t)=I(t)$,当 $t\neq kf$时,系统(2)关于无病 f-周期解 $(\tilde{S}(t),0)$ 的线性化系统为

$$\begin{pmatrix}\frac{\mathrm{d}x}{\mathrm{d}t}\\\frac{\mathrm{d}y}{\mathrm{d}t}\end{pmatrix}=\begin{pmatrix}-k & -\frac{U\tilde{S}(t)}{1+T\tilde{S}(t)}-k\\0 & \frac{U\tilde{S}(t)}{1+T\tilde{S}(t)}-\lambda\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\tag{6}$$

其基解矩阵为

$$\Phi(t)=\begin{pmatrix}\exp(-kt) & h_{12}(t)\\0 & h_{22}(t)\end{pmatrix}$$

其中

$$h_{12}(t)=-\frac{\frac{U\tilde{S}(t)}{1+T\tilde{S}(t)}+k}{\frac{U\tilde{S}(t)}{1+T\tilde{S}(t)}+k-\lambda}\exp\left\{\left(\frac{U\tilde{S}(t)}{1+T\tilde{S}(t)}-\lambda\right)t\right\}$$

$$h_{22}(t)=\exp\left\{\left(\frac{U\tilde{S}(t)}{1+T\tilde{S}(t)}-\lambda\right)t\right\}$$

当 $t=kf$时,由系统(2)得脉冲矩阵

$$\begin{pmatrix}x(t^+)\\y(t^+)\end{pmatrix}=\begin{pmatrix}1-\theta & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$$

于是,系统(6)的单值矩阵为

$$\boldsymbol{M}=\begin{pmatrix}1-\theta & 0\\0 & 1\end{pmatrix}\Phi(f)=\begin{pmatrix}(1-\theta)\exp(-kf) & (1-\theta)h_{12}(f)\\0 & h_{22}(f)\end{pmatrix}$$

显见,系统(6)的两个 Floquet乘子(矩阵 $\boldsymbol{M}$的特征值)分别为

$$\lambda_1=(1-\theta)\exp(-kf)<1$$

$$\lambda_2=h_{22}(f)=\exp\left\{\frac{U(\exp(kf)-1)}{(1+T)(\exp(kf)-1)+\theta}-\lambda\right\}$$

当且仅当 $R'=\frac{U(\exp(kf)-1)}{\lambda((1+T)(\exp(kf)-1)+\theta)}<1$时,$\lambda_2<1$.由 Floquet定理[4],无病 f-周期解 $(\tilde{S}(t),0)$ 局部渐近稳定.证毕.

定理 2　当 $R_0<1$时,系统(2)的无病 f-周期解 $(\tilde{S}(t),0)$ 全局渐近稳定,其中 $R_0=\frac{U(\exp(kf)-1)}{\lambda((\exp(kf)-1)+\theta)}$.

证明　由系统(2)的第一个方程,有

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{\mathrm{d}t}\leqslant \mathrm{k}(1-S(t));t\neq k\mathrm{f} \\ S(t^{+})=(1-\theta)S(t);t=k\mathrm{f}\end{cases} \quad (7)$$

由脉冲比较定理[5],对任意小的正数 X,存在 $N_1\in \mathbf{Z}_+$,使得 $k\geqslant N_1$时,有

$$S(t)\leqslant \tilde{S}(t)+\mathrm{X},k\mathrm{f}\leqslant t<(k+1)\mathrm{f}$$

由式(4)和(5)得到

$$S(t)\leqslant \frac{1-\exp(-\mathrm{k}\mathrm{f})}{1-(1-\theta)\exp(-\mathrm{k}\mathrm{f})}+\mathrm{X}=\frac{S^{*}}{1-\theta}+\mathrm{X}$$

由系统(2)的第二个方程,有

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t}\leqslant \left(\mathrm{U}\left(\frac{S^{*}}{1-\theta}+\mathrm{k}\right)-\lambda\right)I(t)$$

作比较方程

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t}=\left(\mathrm{U}\left(\frac{S^{*}}{1-\theta}+\mathrm{X}\right)-\lambda\right)y(t)$$

当 $R_0=\dfrac{\mathrm{U}S^{*}}{\lambda(1-\theta)}=\dfrac{\mathrm{U}(\exp(\mathrm{k}\mathrm{f})-1)}{\lambda((\exp(\mathrm{k}\mathrm{f})-1)+\theta)}<1$ 时,存在充分小的正数 X,使得 $\mathrm{U}\left(\dfrac{S^{*}}{1-\theta}+\mathrm{X}\right)<\lambda$,于是 $\lim\limits_{t\to\infty}y(t)=0$,从而 $\lim\limits_{t\to\infty}I(t)=0$,即对任意小的正数 X,存在正整数 $N_2\geqslant N_1$,当 $k\geqslant N_2$时,有 $I(t)<\mathrm{X}$.

又由系统(2)的第一个方程,有

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t}\geqslant -\mathrm{U}\mathrm{X}S(t)+\mathrm{k}(1-\mathrm{X}-S(t))$$

作比较方程

$$\begin{cases}\dfrac{\mathrm{d}u(t)}{\mathrm{d}t}=-\mathrm{U}\mathrm{X}u(t)+\mathrm{k}(1-\mathrm{X}-u(t));t\neq k\mathrm{f} \\ u(t^{+})=(1-\theta)u(t);t=k\mathrm{f}\end{cases} \quad (8)$$

则系统(8)在 $k\mathrm{f}\leqslant t<(k+1)\mathrm{f}$内的周期解为

$$\tilde{u}(t)=\frac{\mathrm{k}(1-\mathrm{X})}{\mathrm{U}\mathrm{X}+\mathrm{k}}\left(1-\frac{\theta\exp(-(\mathrm{U}\mathrm{X}+\mathrm{k})(t-k\mathrm{f}))}{1-(1-\theta)\exp(-(\mathrm{U}\mathrm{X}+\mathrm{k})\mathrm{f})}\right)$$

由脉冲比较定理,对任意 $\mathrm{X}>0$,存在正整数 $N_3\geqslant N_2$,使得当 $k\geqslant N_3$时,有

$$\tilde{u}(t)-\mathrm{X}\leqslant S(t)\leqslant \tilde{S}(t)+\mathrm{X},\ k\mathrm{f}\leqslant t<(k+1)\mathrm{f}$$

当 $\mathrm{X}\to 0$时,有 $\tilde{S}(t)-\mathrm{X}\leqslant S(t)\leqslant \tilde{S}(t)+\mathrm{X}$,即 $\lim\limits_{t\to\infty}S(t)=\tilde{S}(t)$.于是,当 $R_0<1$时,系统(2)的 f-周期解 $(\tilde{S}(t),0)$是全局吸引的.由定理 1,$R'<1$时 $(\tilde{S}(t),0)$局部渐近稳定,而 $R'<R_0$,所以当 $R_0<1$时 $(\tilde{S}(t),0)$全局渐近稳定.证毕.

3　正周期解的存在性

由定理 2,系统(2)存在全局渐近稳定的无病周期解 $(\tilde{S}(t),0)$.下面把脉冲周期 f作为分支参数,用分支定理[6]来研究在 $(\tilde{S}(t),0)$附近是否会分支出非平凡的周期解.

为了与分支定理中的记号一致,令 $x_1(t)=S(t)$,$x_2(t)=I(t)$,这时系统(2)变成

$$\left\{\begin{aligned}&\left.\begin{aligned}x_1'(t)&=-\frac{\mathrm{U}x_1(t)x_2(t)}{1+\mathrm{T}x_1(t)}+\mathrm{k}(1-x_1(t)-x_2(t))\triangleq F_1(x_1(t),x_2(t))\\ x_2'(t)&=\frac{\mathrm{U}x_1(t)x_2(t)}{1+\mathrm{T}x_1(t)}-\lambda x_2(t)\triangleq F_2(x_1(t),x_2(t))\end{aligned}\right\}\ t=k\mathrm{f}\\ &\left.\begin{aligned}x_1(t^{+})&=(1-\theta)x_1(t)\triangleq \theta_1(x_1(t),x_2(t))\\ x_2(t^{+})&=x_2(t)\triangleq \theta_2(x_1(t),x_2(t))\end{aligned}\right\}\ t\neq k\mathrm{f}\end{aligned}\right. \quad (9)$$

令 $\mathrm{Y}(t)=(\tilde{S}(t),0)=(\tilde{x}_1(t),0)$,

$$\frac{\partial\mathrm{H}_1(\mathrm{f}_0,x_0)}{\partial x_1}=\exp\left(\int_0^{\mathrm{f}_0}\frac{\partial F_1(\mathrm{Y}(r))}{\partial x_1}\mathrm{d}r\right),$$

$$\frac{\partial\mathrm{H}_2(\mathrm{f}_0,x_0)}{\partial x_2}=\exp\left(\int_0^{\mathrm{f}_0}\frac{\partial F_2(\mathrm{Y}(r))}{\partial x_2}\mathrm{d}r\right),$$

$$\frac{\partial\mathrm{H}_1(\mathrm{f}_0,x_0)}{\partial x_2}=\int_0^{\mathrm{f}_0}\left\{\exp\left(\int_u^{\mathrm{f}_0}\frac{\partial F_1(\mathrm{Y}(r))}{\partial x_1}\mathrm{d}r\right)\times\frac{\partial F_1(\mathrm{Y}(u))}{\partial x_2}\exp\left(\int_0^u\frac{\partial F_2(\mathrm{Y}(r))}{\partial x_2}\mathrm{d}r\right)\right\}\mathrm{d}u$$

经计算得到

$$d_0'=1-\left(\frac{\partial\theta_2}{\partial x_2}\frac{\partial\mathrm{H}_2}{\partial x_2}\right)(\mathrm{f}_0,x_0)=1-\exp\left(\int_0^{\mathrm{f}_0}\left(\frac{\mathrm{U}\tilde{x}_1(r)}{1+\mathrm{T}\tilde{x}_1(r)}-\lambda\right)\mathrm{d}r\right),$$

(f_0是 $d_0'=0$的根)

$$a_0'=1-\left(\frac{\partial\theta_1}{\partial x_1}\frac{\partial\mathrm{H}_1}{\partial x_1}\right)(\mathrm{f}_0,x_0)=1-(1-\theta)\exp(-\mathrm{k}\mathrm{f}_0)>0,$$

$$b_0'=-\left(\frac{\partial\theta_1}{\partial x_1}\frac{\partial\mathrm{H}_1}{\partial x_2}+\frac{\partial\theta_1}{\partial x_2}\frac{\partial\mathrm{H}_2}{\partial x_2}\right)(\mathrm{f}_0,x_0)=-(1-\theta),$$

$$\int_0^{\mathrm{f}_0}\left\{\exp(-\mathrm{k}(\mathrm{f}_0-u))\left(-\frac{\mathrm{U}\tilde{x}_1(u)}{1+\mathrm{T}\tilde{x}_1(u)}-\mathrm{k}\right)\times\exp\left(\int_0^u\left(\frac{\mathrm{U}\tilde{x}_1(r)}{1+\mathrm{T}\tilde{x}_1(r)}-\lambda\right)\mathrm{d}r\right)\right\}\mathrm{d}u>0,$$

$$\frac{\partial^2\mathrm{H}_2(\mathrm{f}_0,x_0)}{\partial x_1\partial x_2}=\int_0^{\mathrm{f}_0}\exp\left(\int_u^{\mathrm{f}_0}\frac{\partial F_2(\mathrm{Y}(r))}{\partial x_2}\mathrm{d}r\right)\times\frac{\partial^2F_2(\mathrm{Y}(u))}{\partial x_1\partial x_2}\exp\left(\int_0^u\frac{\partial F_2(\mathrm{Y}(r))}{\partial x_2}\mathrm{d}r\right)\mathrm{d}u=\int_0^{\mathrm{f}_0}\exp\left(\int_u^{\mathrm{f}_0}\frac{\partial F_2(\mathrm{Y}(r))}{\partial x_2}\mathrm{d}r\right)\frac{\mathrm{U}}{(1+\mathrm{T}\tilde{x}_1(u))^2}\times\exp\left(\int_0^u\frac{\partial F_2(\mathrm{Y}(r))}{\partial x_2}\mathrm{d}r\right)\mathrm{d}u>0,$$

$$\frac{\partial^2 H_2(f_0,x_0)}{\partial x_2^2}=\int_0^{f_0}\exp\left(\int_u^{f_0}\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right)\frac{\partial^2 F_2(Y(u))}{\partial x_2^2}\times\exp\left(\int_0^u\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right)\mathrm{d}u+$$
$$\int_0^{f_0}\left\{\exp\left(\int_u^{f_0}\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right)\frac{\partial^2 F_2(Y(u))}{\partial x_2\partial x_1}\times\int_0^u\left(\exp\left(\int_p^u\frac{\partial F_1(Y(r))}{\partial x_1}\mathrm{d}r\right)\frac{\partial F_1(Y(u))}{\partial x_2}\times\exp\left(\int_0^p\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right)\right)\mathrm{d}p\right\}\mathrm{d}u=$$
$$\int_0^{f_0}\left\{\exp\left(\int_u^{f_0}\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right)\frac{U}{(1+T\tilde{x}_1(u))^2}\times\int_0^u\left(\exp\left(\int_p^u\frac{\partial F_1(Y(r))}{\partial x_1}\mathrm{d}r\right)\times\left(\frac{-U\tilde{x}_1(u)}{1+T\tilde{x}_1(u)}-k\right)\times\exp\left(\int_0^p\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right)\right)\mathrm{d}p\right\}\mathrm{d}u<0,$$

$$\frac{\partial^2 H_2(f_0,x_0)}{\partial f\partial x_2}=\frac{\partial F_2(Y(f_0))}{\partial x_2}\exp\left(\int_0^{f_0}\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right)=\left(\frac{U\tilde{x}_1(f_0)}{1+T\tilde{x}_1(f_0)}-\lambda\right)\exp\left(\int_0^{f_0}\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right),$$

$$\frac{\partial H_1(f_0,x_0)}{\partial f}=\dot{\tilde{x}}_1(f_0)=k\left(1-\frac{(1-\theta)(1-\exp(-kf_0))}{1-(1-\theta)\exp(-kf_0)}\right)\times\exp(-kf_0)>0,$$

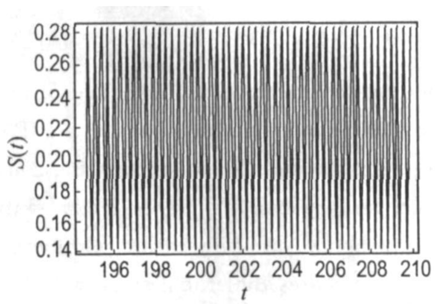
$$B=-\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\left(\frac{\partial H_1(f_0,x_0)}{\partial f}+\frac{\partial H_1(f_0,x_0)}{\partial x_1}\frac{1}{a'_0}\frac{\partial\theta_1}{\partial x_1}\times\frac{\partial H_1(f_0,x_0)}{\partial f}\right)\frac{\partial H_2(f_0,x_0)}{\partial x_2}-\frac{\partial\theta_2}{\partial x_2}\left(\frac{\partial^2 H_2(f_0,x_0)}{\partial f\partial x_2}+\frac{\partial^2 H_2(f_0,x_0)}{\partial x_1\partial x_2}\frac{1}{a'_0}\frac{\partial\theta_1}{\partial x_1}\frac{\partial H_1(f_0,x_0)}{\partial f}\right)=$$
$$-\left(\left(\frac{U\tilde{x}_1(f_0)}{1+T\tilde{x}_1(f_0)}-\lambda\right)\exp\left(\int_0^{f_0}\frac{\partial F_2(Y(r))}{\partial x_2}\mathrm{d}r\right)+\frac{\partial^2 H_2(f_0,x_0)}{\partial x_1\partial x_2}\frac{1}{a'_0}(1-\theta)\frac{\partial H_1(f_0,x_0)}{\partial f}\right),$$

$$C=-2\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\left(-\frac{b'_0}{a'_0}\frac{\partial H_1(f_0,x_0)}{\partial x_1}+\frac{\partial H_1(f_0,x_0)}{\partial x_2}\right)\times\frac{\partial H_2(f_0,x_0)}{\partial x_2}-\frac{\partial^2\theta_2}{\partial x_2^2}\left(\frac{\partial H_2(f_0,x_0)}{\partial x_2}\right)^2+$$
$$2\frac{\partial\theta_2}{\partial x_2}\frac{b'_0}{a'_0}\frac{\partial^2 H_2(f_0,x_0)}{\partial x_2\partial x_1}-\frac{\partial\theta_2}{\partial x_2}\frac{\partial^2 H_2(f_0,x_0)}{\partial x_2^2}=2\frac{b'_0}{a'_0}\frac{\partial^2 H_2(f_0,x_0)}{\partial x_2\partial x_1}-\frac{\partial^2 H_2(f_0,x_0)}{\partial x_2^2}>0$$

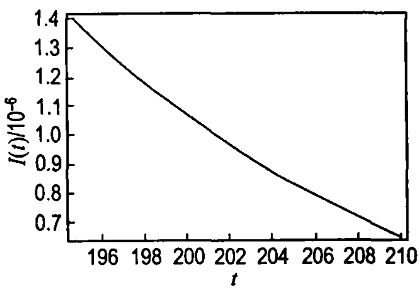
下面判断 B的符号. 令 $f(t)=\dfrac{U\tilde{x}_1(t)}{1+T\tilde{x}_1(t)}-\lambda$, $s'_0=\dfrac{\theta}{1-(1-\theta)\exp(-kf)}$, 有 $f'(t)=\dfrac{Us'_0k\exp(-kt)}{[1+T(1-s'_0\exp(-kt))]^2}>0$,因此 $f(t)$是严格递增的. 又由于 f_0是 $d'_0=0$的根, 有 $\int_0^{f_0}f(t)\mathrm{d}t=f(t)f_0=0$,从而有 $f(f_0)>0$,因此得到 $B<0$(也即有 $BC<0$). 利用分支定理,有下面的结论.

定理 3　在点 f_0处系统(2)发生了超临界分支,即 $f>f_0$并且在 f_0附近,系统(2)存在一个正周期解.

在模型(1)中,设 $U=2$, $T=1$, $k=0.6$, $\lambda=0.4$, $\theta=0.5$, 初始值取 $(S(0),I(0),R(0))=(0.6,0.02,0.375)$. 图 2显示了当 $f=0.3$时 $S(t)$、$I(t)$的时间序列. 可以看到,当 t从 195到 210时, $I(t)\to0$, $S(t)$渐近稳定到一个周期解. 图 3显示了当 $f=0.4$时 $S(t)$、$I(t)$的时间序列. 可以看到,当 t从 260到 280时, $S(t)$、$I(t)$分别渐近稳定到一个周期解. 由上可知,以脉冲周期作为分支参数的分支临界值 f_0在 0.3和 0.4之间.



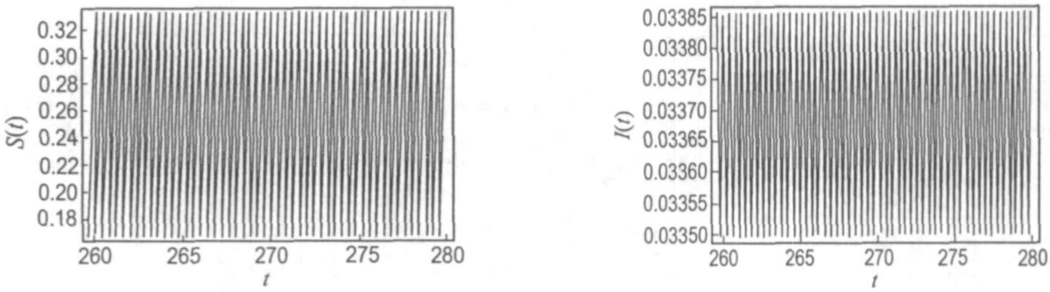
(a) $S(t)$



(b) $I(t)$

图 2　$f=0.3$时的时间序列图

Fig. 2　Time-series when $f=0.3$



(a) $S(t)$ (b) $I(t)$

图 3 $\tau = 0.4$时的时间序列图

Fig. 3 Time-series when $\tau = 0.4$

4 结 论

对生物体进行脉冲免疫接种能有效地防止疾病的产生和控制疾病的蔓延，通常的策略是提高免疫接种率. 在系统(1)中，只要免疫接种率 $\theta > \theta^* := \dfrac{U-\lambda}{\lambda(\exp(k\tau)-1)}$，或者脉冲免疫周期 $\tau < \tau^* := \dfrac{1}{k}\ln\dfrac{U-\lambda+\lambda\theta}{U-\lambda}$ 时，疾病消除；而当 $\tau > \tau_0$ (τ_0 是方程 $d'_{\sigma} = 0$的根)时，疾病会周期性地发生，从而形成地方病.

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Analysis of SIRS model with saturated contact rate and pulse vaccination

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Abstract In order to constitute the reasonable immune vaccination strategy and prevent the infectious diseases from arising and spreading effectively, the dynamical behavior of a SIRS model with saturated contact rate and pulse vaccination is investigated. Using Floquet theory and comparison theorem of impulsive differential equation, the existence and globally asymptotically stability of infection-free periodic solution are proven. Using bifurcation theory, the bifurcation parameter for the existence of positive periodic solution is obtained. The result indicates that the disease will die out if $\theta > \theta^*$ (or $\tau < \tau^*$), whereas the system is uniformly persistent if $\tau > \tau_0$, which means that after some period of time the disease will become endemic.

Key words pulse; vaccination; SIRS model; globally asymptotically stability; periodic solution