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Direct algorithms for constructing high-order conservation laws of nonlinear partial differential equations

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Abstract: The direct algorithms for constructing the conservation laws of nonlinear differential equations are put forward and implemented in software Maple, which are easy for operation and high efficiency. As applications of the algorithms, some higher-dimensional nonlinear differential equations, such as Caudrey-Dodd-Gibbon-Sawada-Kotera equation, Boiti-Leon-Manna-Pempinelli equation and (2+1)-dimensional Burgers equation together with Itô equations are considered. As a result, some new high-order conservation laws of these equations have been obtained. The algorithms can be used to construct more higher order and dimension of conservation laws and be generalized to differential-difference equations.

Key words: conservation law; Caudrey-Dodd-Gibbon-Sawada-Kotera equation; (2+1)-dimensional Burgers equation; Boiti-Leon-Manna-Pempinelli equation; Itô equations

0 Introduction

Nonlinear partial differential equations (NLPDEs) that admit conservation laws arise in many disciplines of the applied sciences, including physical chemistry, fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magnetohydrodynamics, nonlinear optics, and bio-sciences.

Conservation laws are fundamental laws of physics and play important role in mathematical physics. The investigation of conservation laws of the Korteweg de Vries equation was the starting point of the discovery of a number of techniques to solve evolutionary equations (Miura transformation, Lax pair, inverse scattering technique, bi-Hamiltonian structures). The existence of a large number of conservation laws of a PDE (system) is a strong indication of its integrability[1]. The knowledge of conservation laws is useful in the numerical integration of PDEs, for instance, conserved densities aid in the design of numerical solvers for PDEs and controlling numerical errors[2].

There are various methods[3, 4] to compute conservation laws of NLPDEs. A common approach relies on the link between conservation laws and symmetries as stated in Noether's theorem[5, 6]. Most of the algorithmic

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methods[7-10] require the solution of a determining system of PDEs. Hereman[11] presented a direct method for the computation of polynomial conservation laws of polynomial systems of nonlinear partial differential equations in multi-dimensions based on calculus, variational calculus and linear algebra. This paper is to purposely avoid Noether's theorem, pre-knowledge of symmetries, and a Lagrangian formulation. Neither differential forms nor advanced differential-geometric tools are involved. Instead, the two direct algorithms presented only refer to the definition of conservation laws based on differential calculus and linear algebra.

1 Conservation laws

Let Δ be a system of differential equations $\Delta_v(x, u^{(n)})=0$ $(v=1,\cdots, l)$ for m unknown functions $u=(u_1,\cdots,u_m)$ of n independent variables $x=(x_1,\cdots,x_n)$. Here, $u^{(n)}$ denotes the set of all the derivatives of the functions u with respect to x of order not greater than n, including u as the derivative of order zero.

Definition 1 A conserved vector of the system Δ is an n-tuple $\boldsymbol{F}=(F_1(x,u^{(r)})\quad\cdots\quad F_n(x,u^{(r)}))$ for which the divergence $\operatorname{div}\boldsymbol{F}:=\mathrm{D}_iF_i$ vanishes for all solutions of Δ, i. e.

$$\operatorname{div}\boldsymbol{F}\mid_{\Delta=0}=0 \tag{1}$$

where $\mathrm{D}_i=\mathrm{D}_{x_i}$ denotes the operator of total differentiation with respect to the variable x_i, i. e., $\mathrm{D}_i=\partial_{x_i}+u^a_{a,i}\partial_{u^a_a}$. The notation $V|_L$ means that the values of V are considered only on solutions of the system L.

Definition 2 A conserved vector $\boldsymbol{F}$ is called trivial if $F_i=\widehat{F}_i+\breve{F}_i$, where $\widehat{F}_i$ and $\breve{F}_i$ are smooth functions of x and derivatives of u, $\widehat{F}_i$ vanishes on the solutions of Δ and the n-tuple $\widehat{\boldsymbol{F}}=(\widehat{F}_1\quad\widehat{F}_2\quad\cdots\quad\widehat{F}_n)$ is a null divergence (i. e., its divergence vanishes identically).

Definition 3 Two conserved vectors $\boldsymbol{F}$ and $\boldsymbol{F}'$ are called equivalent if the vector-function $\boldsymbol{F}-\boldsymbol{F}'$ is a trivial conserved vector.

Definition 4

$$\operatorname{div}\boldsymbol{F}=\sum_{\mu=1}^{l}\lambda^{\mu}\Delta_{\mu} \tag{2}$$

and the l-tuple $\lambda=(\lambda^1,\lambda^2,\cdots,\lambda^l)$ are called the characteristic form and the characteristic of the conservation law $\operatorname{div}\boldsymbol{F}=0$, respectively.

2 Direct algorithms for constructing conservation laws

2.1 The first direct algorithm

In order to construct conservation laws, one should select the form as follows:

$$\mathrm{D}_tp_1+\mathrm{D}_xp_2=q\Delta \quad \text{or} \quad \mathrm{D}_tp_1+\mathrm{D}_xp_2+\mathrm{D}_yp_3=q\Delta \tag{3}$$

which is just the definition of conservation law and is also the application of the Noether theorem in essence. To construct higher order conservation laws more easily, it is necessary to raise the order of the characteristic $q(x,t,u^{(n)})$ or $q(x,t,y,u^{(n)})$.

Step 1 Assume that p_1,p_2,p_3 and q are functions of $x,t,u^{(n)}$, in order to make p_1,p_2,p_3 and q only contain the derivatives of u of one or two variables, one should increase the order of the derivatives of u in the calculation process gradually.

Step 2 Denote the original equation as $\Delta=0$, and suppose that

$$q\Delta-p_{1t}-p_{2x}=0 \quad \text{or} \quad q\Delta-p_{1t}-p_{2x}-p_{3y}=0 \tag{4}$$

Step 3 Separating the item of u_t and the highest derivatives and setting the coefficients of them to zero yield a simplified system of PDEs.

Step 4 Solve the system by making use of Maple, the conservation laws p_1,p_2,p_3 of the original equation can be obtained,

$$\mathrm{D}_tp_1+\mathrm{D}_xp_2=0 \quad \text{or} \quad \mathrm{D}_tp_1+\mathrm{D}_xp_2+\mathrm{D}_yp_3=0 \tag{5}$$

together with the characteristic q of the corresponding conservation law.

Example 1 The Caudrey-Dodd-Gibbon-Sawada-Kotera equation[12] is as follows:

$$u_t + u_{xxxxx} + 30u_x u_{xx} + 30uu_{xxx} + 180u^2 u_x = 0 \tag{6}$$

Step 1 Assuming $q=q(t,x,u)$, $p_1=p_1(t,x,u)$ and $p_2=p_2(t,x,u,u_x,u_{xx},u_{xxx},u_{xxxx})$.

Step 2 Setting the original equation as $\Delta=0$, then (3) becomes into

$$q(u_t + u_{xxxxx} + 30u_x u_{xx} + 30uu_{xxx} + 180u^2 u_x) - p_{11} - p_{22} - p_{13}u_t - p_{23}u_x - p_{24}u_{xx} - p_{25}u_{xxx} - p_{26}u_{xxxx} - p_{27}u_{xxxxx} = 0 \tag{7}$$

Step 3 Setting $x=X$, $t=T$, $u=U$, $u_x=V$, $u_{xx}=W$, $u_{xxx}=L$, $u_{xxxx}=M$, $u_{xxxxx}=N$ and separating the item of u_t, N yields to

$$q-p_{1U}=q-p_{2M}=0 \tag{8}$$

$$q(30VW+30UL+180U^2V)-p_{1T}-p_{2X}-p_{2U}V-p_{2V}W-p_{2W}L-p_{2L}M=0$$

Step 4 Solving this system , one can get

$$\begin{aligned} p_1 &= u \\ p_2 &= u_{xxxx}+30uu_{xx}+60u^3 \end{aligned} \tag{9}$$

Similarly, if $p_1=p_1(t,x,u,u_x)$ and $p_2=p_1(t,x,u,u_x,u_{xx},u_{xxx},u_{xxxx},u_{xxxxx})$, one can have the fifth-order conservation law $D_t p_1 + D_x p_2 = 0$, where

$$p_1 = F_1(t)u_x + Cu;$$

$$p_2 = F_1(t)(u_{xxxxx}+30u_x u_{xx}+30uu_{xxx}+180u^2u_x) + C(u_{xxxx}+30uu_{xx}+60u^3) - u\int F_1'(t)\mathrm{d}t \tag{10}$$

Example 2 Boiti-Leon-Manna-Pempinelli equation is as follows:

$$u_{yt}+u_{xxxy}-3u_{xx}u_y-3u_{xy}u_x=0 \tag{11}$$

According to the algorithm, the third-order conservation law $D_t p_1 + D_x p_2 + D_y p_3 = 0$ can be obtained, where

$$p_1 = F(x,t)u_y, \qquad p_3 = F(x,t)u_{xxx},$$

$$p_2 = \int^x [F_t(a,t)(-u_y) + 3F(a,t)(u_{xx}u_y+u_{xy}u_x)]\mathrm{d}a \tag{12}$$

2.2 The second algorithm

In order to avoid the complexity of computation in the first algorithm, another one can be proposed. By differentiating the original equation $\Delta=0$ with regard to x, t or y, one can raise the order of the original equation instead of unknown variable $q(t,x,y,u^{(n)})$.

Step 1 Differentiating the original equation $\Delta=0$ with regard to x, t or y yields to a new equation $\Delta_1=0$, and then

$$q\Delta_1 - p_{1t} - p_{2x}=0 \quad \text{or} \quad q\Delta_1 - p_{1t} - p_{2x} - p_{3y}=0 \tag{13}$$

Step 2 Increasing the order of the derivatives of u in the calculation process gradually to make p_1, p_2, p_3 and q only contain the derivatives of u with respect to one or two variables.

Steps 3-5 are the same as the first algorithm.

Example 3 Take 2+1-dimensional Burgers equation for example,

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0 \tag{14}$$

Step 1 Differentiating it once with regard to x yields the following:

$$\Delta_1 = u_{txx} + 3u_x u_{xx} + uu_{xxx} - u_{xxxx} + u_{xyy} = 0$$

Step 2 Assume that $q=q(t,x,y)$, $p_1=p_1(t,x,y,u_{xx})$, $p_2=p_2(t,x,y,u,u_x,u_{xx},u_{xxx})$ and $p_3=p_3(t,x,y,u_{xy})$, then (13) turns to

$$q(u_{txx}+3u_xu_{xx}+uu_{xxx}-u_{xxxx}+u_{xyy}) - p_{11} - p_{14}u_{txx} - p_{22} - p_{24}u_x - p_{25}u_{xx} - p_{26}u_{xxx} - p_{27}u_{xxxx} - p_{33} - p_{34}u_{xyy} = 0 \tag{15}$$

Step 3 Regarding $t,x,y,u,u_x,u_{xx},u_{xy}$ and u_{xxx} as independent variables and substituting $t=T, x=X, y=Y, u=U, u_x=V, u_{xx}=W, u_{xy}=L$ and $u_{xxx}=M$ into Eq. (15).

Step 4 Setting the coefficients of W_T, M_X, L_Y and 1 yields the following:

$$\begin{aligned} &q-p_{14}=q-p_{27}=q-p_{34}=0; \\ &3qVW+qUM-p_{11}-p_{22}-p_{24}V- \\ &p_{25}W-p_{26}M-p_{33}=0 \end{aligned} \tag{16}$$

Step 5 Solving the system with Maple package, the third-order conservation law $D_t p_1 + D_x p_2 + D_y p_3 = 0$ can be derived, where

$$p_1 = u_{xx}\left(\frac{C}{2}x^2 + F_1(t)x + F_2(t)\right);$$

$$p_2 = 2(u - xu_x)F_1'(t) - 2u_xF_2'(t) + (u_{xxx} + uu_{xx} + u_x^2)\left(xF_1(t) + F_2(t) + \frac{C}{2}x^2\right) - C(u_{xx} + uu_x)x - (u_{xx} + uu_x)F_1(t) + Cu_x + \frac{C}{2}u^2;$$

$$p_3 = u_{xy}\left(\frac{C}{2}x^2 + F_1(t)x + F_2(t)\right) \quad (17)$$

Similarly, if one differentiates Eq. (14) twice with regard to x, one can get the fourth-order conservation law $D_tp_1 + D_xp_2 + D_yp_3 = 0$, omitted here.

Example 4　Itô equation[13]

$$u_t = u_{xxx} + 6uu_x + 2vv_x;\ v_t = 2(uv)_x \quad (18)$$

some conservation laws of which has been given by Wolf[10], some new classes of conservation laws will be constructed here.

If one differentiates Eq. (18) once with regard to x, one can get the third-order conservation law $D_tp_1 + D_xp_2 = 0$, where

$$p_1 = F_1(t)u_x + Cxv_x + F_2(t)v_x;$$

$$p_2 = -F_1'(t)u - F_2'(t)v - (u_{xxx} + 6uu_x + 2vv_x)F_1(t) - (2uv_x + 2vu_x)F_2(t) + 2Cu(v - xv_x) - 2Cxvu_x \quad (19)$$

To differentiate Eq. (18) twice with regard to x, one can get the fourth-order conservation law $D_tp_1 + D_xp_2 = 0$, where

$$p_1 = C_1u_{xx} + \frac{C}{2}x^2v_{xx} + F_2(t)xv_{xx} + F_3(t)v_{xx};$$

$$p_2 = F_2'(t)(v - xv_x) - F_3'(t)v_x - C_1(u_{xxxx} + 6uu_{xx} + 2v_x^2 + 2vv_{xx} + 6u_x^2) - 2F_3(t)\times(vu_{xx} + 2u_xv_x + uv_{xx}) - C(uv_{xx} + vu_{xx} + 2u_xv_x)x^2 - 2Cuv + F_2(t)(2vu_x + 2uv_x - 2(uv_{xx} + vu_{xx} + 2u_xv_x)x) + 2C(vu_x + uv_x)x \quad (20)$$

If one differentiates Eq. (18) three times with regard to x, one can get the fifth-order conservation law $D_tp_1 + D_xp_2 = 0$, where

$$p_1 = F_1(t)u_{xxx} + v_{xxx}\left[\frac{C}{6}x^3 + \frac{C}{2}F_2(t)x^2 + xF_3(t) + F_4(t)\right];$$

$$p_2 = -\frac{1}{2}F_2'(t)(x^2v_{xx} - 2xv_x + 2v) - F_1(t)\times(u_{xxxxx} + 2vv_{xxx} + 6uu_{xx} + 18u_xu_{xx} + 6v_xv_{xx}) + F_2(t)[2x(uv_{xx} + 2u_xv_x + vu_{xx}) - x^2(vu_{xxx} + 3v_xu_{xx} + 3u_xv_{xx} + uv_{xxx}) - 2(vu_x + uv_x)] + 2F_3(t)[(uv_{xx} + 2u_xv_x + vu_{xx}) - x(vu_{xxx} + 3v_xu_{xx} + 3u_xv_{xx} + uv_{xxx})] - \left(2F_4(t) + \frac{C}{3}x^3\right)\times(vu_{xxx} + 3v_xu_{xx} + 3u_xv_{xx} + uv_{xxx}) + F_3'(t)(v_x - xv_{xx}) - F_4'(t)v_{xx} - F_1'(t)u_{xx} + Cx^2(uv_{xx} + 2u_xv_x + vu_{xx}) - 2Cx(vu_x + uv_x) + 2Cuv \quad (21)$$

Remark 1　The symbols $F_i(t)\,(i=1,\cdots,4)$ and $F(x,t)$ appearing in the above conservation laws are all arbitrary differential functions and C, C_1 are arbitrary constants.

3　Conclusion

Two kinds of algorithms for constructing high-order conservation laws of NLPDEs have been put forward. As applications, many new high-order conservation laws of a large number of NLPDEs have been obtained, such as the Caudrey-Dodd-Gibbon-Sawada-Kotera equation, (2+1)-dimensional Burgers equation and Boiti-Leon-Manna-Pempinelli equation. In fact, the algorithms are very efficient and easy to be extended to a variety of NLPDEs and even differential-difference equations.

References:

[1] ABLOWITZ M J, CLARKSON P A. **Solitons, Nonlinear Evolution Equations and Inverse Scattering** [M]. Cambridge: Cambridge University Press, 1991

[2] SANZ-SERNA J M. An explicit finite-difference scheme with exact conservation properties [J]. **Journal of Computational Physics**, 1982, **47**: 199-210

[3] HEREMAN W, COLAGROSSO M, SAYERS R, *et al*. **Differential Equations with Symbolic Computation** [M]. Basel: Birkhäuser Verlag, 2005

[4] ANDERSON I M. **The Variational Bicomplex** [M]. Logan:Department of Mathematics and Statistics at Utah State University, 2004

[5] OLVER P J. **Applications of Lie Groups to Differential Equations** [M]. New York: Springer Verlag, 1993

[6] HEREMAN W, ADAMS P J, EKLUND H L, *et al*. **Direct Methods and Symbolic Software for Conservation Laws of Nonlinear Equations , Advances in Nonlinear Waves and Symbolic Computation** [M]. New York:Nova Science Publishers, 2008

[7] IBRAGIMOV N H, KOLSRUD T. Lagrangian approach to evolution equations: Symmetries and conservation laws [J]. **Nonlinear Dynamics**, 2004, **36**:9-40

[8] ANCO S C, BLUMAN G. Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications [J]. **European Journal of Applied Mathematics**, 2002, **13**:545-566

[9] KARA A H, MAHOMED F M. Relationship between symmetries and conservation laws [J]. **International Journal of Theoretical Physics**, 2000, **39**(1):23-40

[10] WOLF T. A comparison of four approaches to the calculation of conservation laws [J]. **European Journal of Applied Mathematics**, 2002, **13**:129-152

[11] HEREMAN W. Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions [J]. **International Journal of Quantum Chemistry**, 2006, **106**(1):278-299

[12] ZWILLINGER D. **Handbook of Differential Equations** [M]. Boston:Academic Press, 1997

[13] ITÔ M. An extension of nonlinear evolution equations of the KdV ($mKdV$) type to higher orders [J]. **Journal of the Physical Society of Japan**, 1980, **49**:771-778

构造非线性偏微分方程高阶守恒律的直接法

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摘要： 提出了构造非线性偏微分方程高阶守恒律的直接法并在 Maple 上实现，算法易操作，效率高. 作为算法的应用，考虑了许多高维非线性偏微分方程，如 Caudrey-Dodd-Gibbon-Sawada-Kotera 方程、Boiti-Leon-Manna-Pempinelli 方程和(2+1)-维 Burgers 方程以及 Itô 方程组，得到了它们的新的高阶守恒律. 该算法还可用于构造更高维更高阶的守恒律，亦可推广至微分-差分方程(组).

关键词： 守恒律；Caudrey-Dodd-Gibbon-Sawada-Kotera 方程；(2+1)-维 Burgers 方程；Boiti-Leon-Manna-Pempinelli 方程；Itô 方程组

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