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非线性半定规划的雅可比唯一性定理

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摘要： 提出了非线性半定规划的雅可比唯一性条件，证明在这一条件下，刻画 KKT 条件的映射在 KKT 点处导数是非奇异的. 在雅可比唯一性条件下，证明了非线性半定规划的稳定性定理并建立了下层为非线性半定优化问题的一类特殊双层规划的必要性最优条件.

关键词： 半定规划；雅可比唯一性条件；稳定性；双层规划

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0 引　言

最优化问题的扰动分析是非常重要的专题，在数值算法实现的稳健性分析和双层规划的理论研究中起着非常重要的作用. 目前，扰动分析的研究已经取得了丰富的进展，比如近年来国际优化领域出版了关于变分分析、扰动分析、非光滑方程和互补与变分不等式的著名专著[1-4]，在这些专著中最优化的扰动理论都不同程度地被给予关注. 文献[5]详细介绍了非线性规划的扰动分析结果，文献[2]详细介绍了一般最优化问题的扰动分析结果.

追溯到扰动分析的早期工作，讨论的问题非常特殊，如讨论问题的函数是二次连续可微的，扰动后的函数关于决策变量和扰动参数也是二次连续可微的，在此情况下，扰动问题解的存在性、连续性和微分性质. Fiacco 等[6] 在 1968 年对非线性规划在这种情况的扰动分析给出讨论，提出了著名的雅可比唯一性条件(Jacobian uniqueness conditions). 对非线性半定规划而言，类似的雅可比唯一性条件是什么样的条件，由此条件出发得到什么样的稳定性理论，还没有文献涉及，本文讨论这些问题.

1 雅可比唯一性定理

考虑非线性半定规划问题：

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{G}(\boldsymbol{x}) \in \boldsymbol{S}_{-}^{p} \\ & \boldsymbol{x} \in \boldsymbol{R}^{n} \end{aligned} \tag{1}$$

其中 $f:\boldsymbol{R}^n \to \mathbf{R}$ 与 $\boldsymbol{G}:\boldsymbol{R}^n \to \boldsymbol{S}^p$ 是二次连续可微函数和映射. 式(1) 的 Lagrange 函数定义为

$$L(\boldsymbol{x},\boldsymbol{Y}) = f(\boldsymbol{x}) + \langle \boldsymbol{Y},\boldsymbol{G}(\boldsymbol{x})\rangle;(\boldsymbol{x},\boldsymbol{Y}) \in \boldsymbol{R}^n \times \boldsymbol{S}^p$$

称 $(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})$ 为式(1) 的稳定点，如果

$$\mathrm{D}_x L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}}) = \boldsymbol{0};\boldsymbol{0} \geq \boldsymbol{G}(\bar{\boldsymbol{x}}) \perp \bar{\boldsymbol{Y}} \geq \boldsymbol{0}$$

式(1) 在稳定点 $\bar{\boldsymbol{x}}$ 处的临界锥 $\boldsymbol{C}(\bar{\boldsymbol{x}})$ 定义为

$$\boldsymbol{C}(\bar{\boldsymbol{x}})=\{\boldsymbol{d}\in\boldsymbol{R}^n:\mathrm{D}f(\bar{\boldsymbol{x}})\boldsymbol{d}\leqslant 0,\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\boldsymbol{d}\in\boldsymbol{T}_{\boldsymbol{S}_{-}^{p}}(\boldsymbol{G}(\bar{\boldsymbol{x}}))\}$$

设 $\bar{\boldsymbol{x}}$ 是可行点，所谓雅可比唯一性条件是指如下的 4 个条件成立：

(1) 存在 $\bar{\boldsymbol{Y}} \in \boldsymbol{S}^p$ 满足

$$\mathrm{D}_x L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}}) = \boldsymbol{0};\ \boldsymbol{0} \geq \boldsymbol{G}(\bar{\boldsymbol{x}}) \perp \bar{\boldsymbol{Y}} \geq \boldsymbol{0}$$

(2) 约束非退化条件在 $\bar{\boldsymbol{x}}$ 处成立，即

$$\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\boldsymbol{R}^n + \operatorname{lin} \boldsymbol{T}_{\boldsymbol{S}_{-}^{p}}(\boldsymbol{G}(\bar{\boldsymbol{x}})) = \boldsymbol{S}^p$$

(3) 严格互补条件成立，即

$$\lambda_i(\boldsymbol{G}(\bar{\boldsymbol{x}}) + \bar{\boldsymbol{Y}}) \neq 0;\ \forall i = 1,2,\cdots,p$$

(4) 二阶充分条件成立，即

$$\sup_{\boldsymbol{Y}\in\boldsymbol{\Lambda}(\bar{\boldsymbol{x}})} \{\mathrm{D}_{xx}^2 L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\boldsymbol{d},\boldsymbol{d}) + \boldsymbol{d}^{\mathrm{T}}\boldsymbol{H}(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})\boldsymbol{d}\} > 0;$$
$$\forall \boldsymbol{d} \in \boldsymbol{C}(\bar{\boldsymbol{x}})\backslash\{0\}$$

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其中

$$\boldsymbol{H}(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})=\left(-2\left\langle\bar{\boldsymbol{Y}},\frac{\partial\boldsymbol{G}(\bar{\boldsymbol{x}})}{\partial x_i}\boldsymbol{G}(\bar{\boldsymbol{x}})^{\dagger}\frac{\partial\boldsymbol{G}(\bar{\boldsymbol{x}})}{\partial x_j}\right\rangle\right)$$

$\boldsymbol{G}(\bar{\boldsymbol{x}})^{\dagger}$ 是矩阵 $\boldsymbol{G}(\bar{\boldsymbol{x}})$ 的广义逆，集合 $\boldsymbol{\Lambda}(\bar{\boldsymbol{x}})$ 是在 $\bar{\boldsymbol{x}}$ 处的 Lagrange 乘子集合：

$$\boldsymbol{\Lambda}(\bar{\boldsymbol{x}})=\{\boldsymbol{Y}\in\boldsymbol{S}^p:\mathrm{D}_xL(\bar{\boldsymbol{x}},\boldsymbol{Y})=\boldsymbol{0},\boldsymbol{Y}\in\boldsymbol{N}_{\boldsymbol{S}_-^p}(\boldsymbol{G}(\bar{\boldsymbol{x}}))\}$$

定理 1　设 $f:\boldsymbol{R}^n\rightarrow\mathbf{R}$ 与 $\boldsymbol{G}:\boldsymbol{R}^n\rightarrow\boldsymbol{S}_-^p$ 是二次连续可微函数和映射，$\boldsymbol{\Phi}$ 是式(1)的可行集合，$\bar{\boldsymbol{x}}\in\boldsymbol{\Phi}$ 满足条件(1)～(4)，则映射

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{Y})=\begin{pmatrix}\mathrm{D}_xL(\boldsymbol{x},\boldsymbol{Y})\\ \Pi_{\boldsymbol{S}_+^p}(\boldsymbol{Y}+\boldsymbol{G}(\boldsymbol{x}))-\boldsymbol{Y}\end{pmatrix}$$

在 $(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})$ 处的导数是非奇异的.

证明　由条件(3)可得矩阵 $\bar{\boldsymbol{A}}=\bar{\boldsymbol{Y}}+\boldsymbol{G}(\bar{\boldsymbol{x}})$ 是非奇异的，因此有 $\Pi_{\boldsymbol{S}_+^p}$ 在 $\bar{\boldsymbol{A}}$ 处是可微的，$\boldsymbol{F}$ 在 $(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})$ 处是可微的. 设 $\bar{\boldsymbol{A}}$ 具有如下的谱分解：

$$\bar{\boldsymbol{A}}=\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{\mathrm{T}}$$

其中 $\boldsymbol{\Lambda}=\mathbf{diag}\{\lambda_1,\cdots,\lambda_p\}$，$\lambda_1\geqslant\cdots\geqslant\lambda_p$ 是 $\bar{\boldsymbol{A}}$ 的 p 个特征值，$\boldsymbol{P}\in\boldsymbol{R}^{p\times p}$ 是正交矩阵，$\boldsymbol{P}=(q_1\ \ \cdots\ \ q_p)$，则映射 $\Pi_{\boldsymbol{S}_+^p}$ 在 $\bar{\boldsymbol{A}}$ 处沿 $\boldsymbol{H}\in\boldsymbol{S}^p$ 的方向导数为

$$\Pi'_{\boldsymbol{S}_+^p}(\bar{\boldsymbol{A}};\boldsymbol{H})=\boldsymbol{P}(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{P}\circ\boldsymbol{\Omega})\boldsymbol{P}^{\mathrm{T}}\tag{2}$$

其中 $\circ$ 为矩阵的 Hadamard 乘积运算，$\boldsymbol{\Omega}\in\boldsymbol{S}^p$ 的元素 Ω_{ij} 定义为

$$\Omega_{ij}=\frac{[\lambda_i]_++[\lambda_j]_+}{|\lambda_i|+|\lambda_j|};\ i,j=1,2,\cdots,p\tag{3}$$

于是，映射 $\boldsymbol{F}$ 在 $(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})$ 处沿 $(\Delta\boldsymbol{x},\Delta\boldsymbol{Y})$ 的方向导数为

$$\mathrm{D}\boldsymbol{F}(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{Y})=\begin{pmatrix}\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})\Delta\boldsymbol{x}+\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})^*\Delta\boldsymbol{Y}\\ \boldsymbol{P}(\boldsymbol{P}^{\mathrm{T}}(\Delta\boldsymbol{Y}+\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x})\boldsymbol{P}\circ\boldsymbol{\Omega})\boldsymbol{P}^{\mathrm{T}}-\Delta\boldsymbol{Y}\end{pmatrix}$$

由 $\mathrm{D}\boldsymbol{F}(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{Y})=\boldsymbol{0}$ 得

$$\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})\Delta\boldsymbol{x}+\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})^*\Delta\boldsymbol{Y}=\boldsymbol{0}\tag{4}$$

$$\boldsymbol{P}^{\mathrm{T}}\Delta\boldsymbol{Y}\boldsymbol{P}\circ(\mathbf{1}_p\mathbf{1}_p^{\mathrm{T}}-\boldsymbol{\Omega})=\boldsymbol{P}^{\mathrm{T}}(\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x})\boldsymbol{P}\circ\boldsymbol{\Omega}\tag{5}$$

引入指标集合

$$\alpha=\{i:\lambda_i(\bar{\boldsymbol{A}})>0\},\ \gamma=\{i:\lambda_i(\bar{\boldsymbol{A}})<0\}$$

则

$$\boldsymbol{G}(\bar{\boldsymbol{x}})=\boldsymbol{P}\begin{pmatrix}\boldsymbol{0}&\\&\boldsymbol{\Lambda}_{\gamma}\end{pmatrix}\boldsymbol{P}^{\mathrm{T}},\ \bar{\boldsymbol{Y}}=\boldsymbol{P}\begin{pmatrix}\boldsymbol{\Lambda}_{\alpha}&\\&\boldsymbol{0}\end{pmatrix}\boldsymbol{P}^{\mathrm{T}}$$

记 $\boldsymbol{P}_{\alpha}=(p_i:i\in\alpha)$，$\boldsymbol{P}_{\gamma}=(p_i:i\in\gamma)$. 由于严格互补条件成立，临界锥 $\boldsymbol{C}(\bar{\boldsymbol{x}})$ 可以表示为

$$\boldsymbol{C}(\bar{\boldsymbol{x}})=\{\boldsymbol{d}\in\boldsymbol{R}^n:\boldsymbol{P}_{\alpha}^{\mathrm{T}}\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\boldsymbol{d}\boldsymbol{P}_{\alpha}=\boldsymbol{0}\}\tag{6}$$

把 $\boldsymbol{\Omega}$ 表示为

$$\boldsymbol{\Omega}=\begin{pmatrix}\Omega_{\alpha\alpha}&\Omega_{\alpha\gamma}\\ \Omega_{\gamma\alpha}&\Omega_{\gamma\gamma}\end{pmatrix}\tag{7}$$

则 $\Omega_{\alpha\alpha}=\mathbf{1}_{|\alpha|}\mathbf{1}_{|\alpha|}^{\mathrm{T}}$，$\Omega_{\gamma\gamma}=\mathbf{0}_{|\gamma|\times|\gamma|}$，

$$\Omega_{ij}=\frac{\lambda_i}{\lambda_i+|\lambda_j|};\ i\in\alpha,j\in\gamma\tag{8}$$

由式(5)可得

$$\begin{aligned}&\boldsymbol{q}_i^{\mathrm{T}}\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x}\boldsymbol{q}_j=\boldsymbol{0},\ (i,j)\in\alpha\times\alpha;\\ &\boldsymbol{q}_i^{\mathrm{T}}\Delta\boldsymbol{Y}\boldsymbol{q}_j=\frac{\Omega_{ij}}{1-\Omega_{ij}}\boldsymbol{q}_i^{\mathrm{T}}\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x}\boldsymbol{q}_j=\boldsymbol{0},\ (i,j)\in\alpha\times\gamma;\\ &\boldsymbol{q}_i^{\mathrm{T}}\Delta\boldsymbol{Y}\boldsymbol{q}_j=\frac{\lambda_i}{|\lambda_j|}\boldsymbol{q}_i^{\mathrm{T}}\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x}\boldsymbol{q}_j,\ (i,j)\in\alpha\times\gamma;\\ &\boldsymbol{q}_i^{\mathrm{T}}\Delta\boldsymbol{Y}\boldsymbol{q}_j=\boldsymbol{0},\ (i,j)\in\gamma\times\gamma\end{aligned}\tag{9}$$

用 $\Delta\boldsymbol{x}$ 与式(4)两边的向量做内积，并由式(9)可得

$$\begin{aligned}0=&\langle\Delta\boldsymbol{x},\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})\Delta\boldsymbol{x}+\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})^*\Delta\boldsymbol{Y}\rangle=\\ &\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{x})+\langle\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x},\Delta\boldsymbol{Y}\rangle=\\ &\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{x})+\\ &2\sum_{i\in\alpha,j\in\gamma}\frac{\lambda_i}{|\lambda_j|}[\boldsymbol{q}_i^{\mathrm{T}}(\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x})\boldsymbol{q}_j]^2=\\ &\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{x})-\\ &2\sum_{i\in\alpha,j\in\gamma}\frac{\lambda_i}{\lambda_j}[\boldsymbol{q}_i^{\mathrm{T}}(\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x})\boldsymbol{q}_j]^2=\\ &\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{x})-2\sum_{i,j=1}^{p}(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{Y}\boldsymbol{P})_{ij}\times\\ &(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{G}(\bar{\boldsymbol{x}})^{\dagger}\boldsymbol{P})_{ij}(\boldsymbol{P}^{\mathrm{T}}(\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\Delta\boldsymbol{x})\boldsymbol{P})_{ij}^2=\\ &\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{x})-\\ &2\sum_{s,t=1}^{n}\left\langle\boldsymbol{P}^{\mathrm{T}}\bar{\boldsymbol{Y}}\boldsymbol{P},\boldsymbol{P}^{\mathrm{T}}\frac{\partial\boldsymbol{G}(\bar{\boldsymbol{x}})}{\partial x_s}\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{G}(\bar{\boldsymbol{x}})^{\dagger}\times\right.\\ &\left.\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}\frac{\partial\boldsymbol{G}(\bar{\boldsymbol{x}})}{\partial x_t}\boldsymbol{P}^{\mathrm{T}}\right\rangle\Delta x_s\Delta x_t=\\ &\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{x})-2\sum_{s,t=1}^{n}\left\langle\bar{\boldsymbol{Y}},\frac{\partial\boldsymbol{G}(\bar{\boldsymbol{x}})}{\partial x_s}\times\right.\\ &\left.\boldsymbol{G}(\bar{\boldsymbol{x}})^{\dagger}\frac{\partial\boldsymbol{G}(\bar{\boldsymbol{x}})}{\partial x_t}\right\rangle\Delta x_s\Delta x_t=\\ &\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{x})+\langle\Delta\boldsymbol{x},\boldsymbol{H}(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})\Delta\boldsymbol{x}\rangle\end{aligned}$$

即

$$\mathrm{D}_{xx}^2L(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})(\Delta\boldsymbol{x},\Delta\boldsymbol{x})+\langle\Delta\boldsymbol{x},\boldsymbol{H}(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})\Delta\boldsymbol{x}\rangle=0\tag{10}$$

根据在条件(3)成立的前提下临界锥的表达式(6),式(9)的第一式意味着 $\Delta\boldsymbol{x}\in\boldsymbol{C}(\bar{\boldsymbol{x}})$,因此由二阶条件和式(10)可推出 $\Delta\boldsymbol{x}=\boldsymbol{0}$. 由式(4)可得

$$\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})^{*}\Delta\boldsymbol{Y}=\boldsymbol{0}$$

由此结合 $\boldsymbol{P}_{\alpha}^{\mathrm{T}}\Delta\boldsymbol{Y}\boldsymbol{P}_{\gamma}=\boldsymbol{0}$ 与 $\boldsymbol{P}_{\gamma}^{\mathrm{T}}\Delta\boldsymbol{Y}\boldsymbol{P}_{\gamma}=\boldsymbol{0}$ 以及约束非退化条件(2),得到 $\boldsymbol{P}_{\alpha}^{\mathrm{T}}\Delta\boldsymbol{Y}\boldsymbol{P}_{\alpha}=\boldsymbol{0}$,于是得到 $\Delta\boldsymbol{Y}=\boldsymbol{0}$. 证毕.

在雅可比唯一性条件成立的前提下,可以进行式(1)的稳定性分析.

命题 1 考虑如下的扰动问题:

$$\begin{aligned}&\min\quad \bar{f}(\boldsymbol{x},\boldsymbol{u})\\&\text{s. t.}\quad \bar{\boldsymbol{G}}(\boldsymbol{x},\boldsymbol{u})\in\boldsymbol{S}_{-}^{p}\\&\qquad\ \ \boldsymbol{x}\in\boldsymbol{R}^{n}\end{aligned}\tag{11}$$

其中 $\bar{f}:\boldsymbol{R}^{n}\times\boldsymbol{R}^{m}\to\mathrm{R}$ 与 $\bar{\boldsymbol{G}}:\boldsymbol{R}^{n}\times\boldsymbol{R}^{m}\to\boldsymbol{S}^{p}$ 是二次连续可微函数和映射,$\bar{f}(\boldsymbol{x},\boldsymbol{0})=f(\boldsymbol{x})$,$\bar{\boldsymbol{G}}(\boldsymbol{x},\boldsymbol{0})=\boldsymbol{G}(\boldsymbol{x})$. 如果雅可比唯一性条件(1)~(4)成立,则存在 $\varepsilon>0$ 与唯一的二次连续可微映射 $\boldsymbol{x}:\boldsymbol{B}(\boldsymbol{0},\varepsilon)\to\boldsymbol{B}(\bar{\boldsymbol{x}},\varepsilon)$,满足对任意的 $\boldsymbol{u}\in\boldsymbol{B}(\boldsymbol{0},\varepsilon)$,$\boldsymbol{x}(\boldsymbol{u})$ 是式(11)满足二阶增长条件的局部极小点,其中 $\boldsymbol{B}(\bar{\boldsymbol{x}},\varepsilon)$ 表示以 $\bar{\boldsymbol{x}}$ 为球心,ε 为半径的开球,$\boldsymbol{B}(\boldsymbol{0},\varepsilon)$ 也有类似的定义.

证明 定义映射

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{Y})=\begin{pmatrix}\mathrm{D}_{x}L_{0}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{Y})\\ \Pi_{\boldsymbol{S}_{+}^{p}}(\boldsymbol{Y}+\bar{\boldsymbol{G}}(\boldsymbol{x},\boldsymbol{u}))-\boldsymbol{Y}\end{pmatrix}$$

其中

$$L_{0}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{Y})=\bar{f}(\boldsymbol{x},\boldsymbol{u})+\langle\boldsymbol{Y},\bar{\boldsymbol{G}}(\boldsymbol{x},\boldsymbol{u})\rangle$$

是式(11)的 Lagrange 函数,显然有 $\boldsymbol{H}(\bar{\boldsymbol{x}},\boldsymbol{0},\bar{\boldsymbol{Y}})=\boldsymbol{0}$,$\mathrm{D}_{x,Y}\boldsymbol{H}(\bar{\boldsymbol{x}},\boldsymbol{0},\bar{\boldsymbol{Y}})=\mathrm{D}\boldsymbol{F}(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})$ 是映上的. 根据隐函数定理,则存在 $\varepsilon>0$ 与唯一的二次连续可微映射 $(\boldsymbol{x},\boldsymbol{Y}):\boldsymbol{B}(\boldsymbol{0},\varepsilon)\to\boldsymbol{B}(\bar{\boldsymbol{x}},\varepsilon)$,满足 $(\boldsymbol{x}(\boldsymbol{0}),\boldsymbol{Y}(\boldsymbol{0}))=(\bar{\boldsymbol{x}},\bar{\boldsymbol{Y}})$. 对任意的 $\boldsymbol{u}\in\boldsymbol{B}(\boldsymbol{0},\varepsilon)$,$\boldsymbol{H}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u},\boldsymbol{Y}(\boldsymbol{u}))=\boldsymbol{0}$,这表明 $(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{Y}(\boldsymbol{u}))$ 是式(11)的 KKT 点,即

$$\mathrm{D}_{x}L_{0}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u},\boldsymbol{Y}(\boldsymbol{u}))=\boldsymbol{0};\ \boldsymbol{0}\succeq\bar{\boldsymbol{G}}(\boldsymbol{x}(\boldsymbol{u}))\perp\boldsymbol{Y}(\boldsymbol{u})\succeq\boldsymbol{0}$$

由 $(\boldsymbol{x}(\cdot),\boldsymbol{Y}(\cdot))$ 的连续性,对 $\boldsymbol{u}\in\boldsymbol{B}(\boldsymbol{0},\varepsilon)$,式(11)在 $\boldsymbol{x}(\boldsymbol{u})$ 处的约束非退化条件成立,严格互补条件 $\lambda_{i}(\bar{\boldsymbol{G}}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u})+\boldsymbol{Y}(\boldsymbol{u}))\neq 0$,$\forall i$ 成立.

由于 $\boldsymbol{C}(\bar{\boldsymbol{x}})=\{\boldsymbol{d}:\langle\bar{\boldsymbol{Y}},\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\boldsymbol{d}\rangle=\boldsymbol{0}\}$,

$$\boldsymbol{C}(\boldsymbol{x}(\boldsymbol{u}))=\{\boldsymbol{d}:\langle\boldsymbol{Y}(\boldsymbol{u}),\mathrm{D}\bar{\boldsymbol{G}}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u})\boldsymbol{d}\rangle=\boldsymbol{0}\}$$

在 $\boldsymbol{u}=\boldsymbol{0}$ 处连续(在变分分析的集值映射连续的意义下)以及

$$\boldsymbol{H}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{Y}(\boldsymbol{u}))=\left(-2\left\langle\boldsymbol{Y}(\boldsymbol{u}),\frac{\partial\bar{\boldsymbol{G}}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u})}{\partial x_{i}}\times\right.\right.$$
$$\left.\left.\bar{\boldsymbol{G}}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u})^{\dagger}\frac{\partial\bar{\boldsymbol{G}}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u})}{\partial x_{j}}\right\rangle\right)$$

在 $\boldsymbol{u}=\boldsymbol{0}$ 处的连续性,对充分小的 $\varepsilon>0$,$\boldsymbol{u}\in\boldsymbol{B}(\boldsymbol{0},\varepsilon)$ 时,

$$\begin{gathered}\sup_{\boldsymbol{Y}\in\boldsymbol{\Lambda}(\boldsymbol{x}(\boldsymbol{u}))}\{\mathrm{D}_{xx}^{2}L_{0}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u},\boldsymbol{Y}(\boldsymbol{u}))(\boldsymbol{d},\boldsymbol{d})+\\ \boldsymbol{d}^{\mathrm{T}}\boldsymbol{H}(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{Y}(\boldsymbol{u}))\boldsymbol{d}\}>0;\\ \forall\,\boldsymbol{d}\in\boldsymbol{C}(\boldsymbol{x}(\boldsymbol{u}))\backslash\{0\}\end{gathered}$$

即在 $(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{Y}(\boldsymbol{u}))$ 处,式(11)的二阶充分最优性条件成立,因此 $\boldsymbol{x}(\boldsymbol{u})$ 是式(11)满足二阶增长条件的局部极小点.

作为命题 1 的应用,考虑扰动问题:

$$\begin{aligned}&\min\quad f(\boldsymbol{x})\\&\text{s. t.}\quad \boldsymbol{G}(\boldsymbol{x})+\boldsymbol{Z}\in\boldsymbol{S}_{-}^{p}\\&\qquad\ \ \boldsymbol{x}\in\boldsymbol{R}^{n}\end{aligned}\tag{12}$$

其中 $\boldsymbol{Z}\in\boldsymbol{S}^{p}$. 式(12)的最优值函数被称为扰动函数,记为 $\boldsymbol{v}(\boldsymbol{Z})$.

定理 2 设 $f:\boldsymbol{R}^{n}\to\mathrm{R}$ 与 $\boldsymbol{G}:\boldsymbol{R}^{n}\to\boldsymbol{S}_{-}^{p}$ 是二次连续可微函数和映射,$\bar{\boldsymbol{x}}\in\boldsymbol{\Phi}$ 满足条件(1)~(4),则

$$\mathrm{D}\boldsymbol{v}(\boldsymbol{0})\boldsymbol{H}=-\langle\bar{\boldsymbol{Y}},\boldsymbol{H}\rangle;\ \forall\,\boldsymbol{H}\in\boldsymbol{S}^{p}\tag{13}$$

证明 由命题 1,存在 $\varepsilon>0$,唯一的连续可微映射 $(\boldsymbol{x},\boldsymbol{Y}):\boldsymbol{B}(\boldsymbol{0},\varepsilon)\to\boldsymbol{B}(\bar{\boldsymbol{x}},\varepsilon)\times\boldsymbol{B}(\bar{\boldsymbol{Y}},\varepsilon)$,满足对任意的 $\boldsymbol{Z}\in\boldsymbol{B}(\boldsymbol{0},\varepsilon)$,$(\boldsymbol{x}(\boldsymbol{Z}),\boldsymbol{Y}(\boldsymbol{Z}))$ 满足式(12)的 KKT 条件,即

$$\begin{aligned}&\mathrm{D}_{x}L_{0}(\boldsymbol{x}(\boldsymbol{Z}),\boldsymbol{Z},\boldsymbol{Y}(\boldsymbol{Z}))=\boldsymbol{0}\\&\Pi_{\boldsymbol{S}_{+}^{p}}(\boldsymbol{Y}(\boldsymbol{Z})+\boldsymbol{G}(\boldsymbol{x}(\boldsymbol{Z}))+\boldsymbol{Z})-\boldsymbol{Y}(\boldsymbol{Z})=\boldsymbol{0}\end{aligned}\tag{14}$$

其中

$$L_{0}(\boldsymbol{x},\boldsymbol{Z},\boldsymbol{Y})=f(\boldsymbol{x})+\langle\boldsymbol{Y},\boldsymbol{G}(\boldsymbol{x})+\boldsymbol{Z}\rangle$$

由式(14)的第二式得

$$\Pi'_{\boldsymbol{S}_{+}^{p}}(\mathrm{D}\boldsymbol{Y}(\boldsymbol{0})\boldsymbol{H}+\mathrm{D}\boldsymbol{G}(\boldsymbol{x}(\boldsymbol{0}))\boldsymbol{H}+\boldsymbol{H})-\mathrm{D}\boldsymbol{Y}(\boldsymbol{0})\boldsymbol{H}=\boldsymbol{0}$$

即

$$\boldsymbol{P}^{\mathrm{T}}\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\mathrm{D}\boldsymbol{x}(\boldsymbol{0})\boldsymbol{H}\boldsymbol{P}\circ\boldsymbol{\Omega}-\boldsymbol{P}^{\mathrm{T}}\mathrm{D}\boldsymbol{Y}(\boldsymbol{0})\boldsymbol{H}\boldsymbol{P}\circ(\boldsymbol{1}_{p}\boldsymbol{1}_{p}^{\mathrm{T}}-\boldsymbol{\Omega})+\boldsymbol{P}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{P}\circ\boldsymbol{\Omega}=\boldsymbol{0}\tag{15}$$

由于 $(\boldsymbol{1}_{p}\boldsymbol{1}_{p}^{\mathrm{T}}-\boldsymbol{\Omega})_{\alpha\alpha}=\boldsymbol{0}_{|\alpha|\times|\alpha|}$,$\Omega_{\alpha\alpha}=\boldsymbol{1}_{|\alpha|}\boldsymbol{1}_{|\alpha|}{}^{\mathrm{T}}$,由式

(15) 得

$$(\boldsymbol{P}^{\mathrm T}\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\mathrm{D}\boldsymbol{x}(\boldsymbol{0})\boldsymbol{H}\boldsymbol{P})_{\alpha\alpha}=(\boldsymbol{P}^{\mathrm T}\boldsymbol{H}\boldsymbol{P})_{\alpha\alpha} \tag{16}$$

根据

$$\nabla f(\bar{\boldsymbol{x}})=-\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})^{*}\bar{\boldsymbol{Y}}$$

和式(16)得

$$\begin{aligned}\mathrm{D}\boldsymbol{v}(\boldsymbol{0})\boldsymbol{H}&=\nabla f(\boldsymbol{x}(\boldsymbol{0}))^{\mathrm T}\mathrm{D}\boldsymbol{x}(\boldsymbol{0})\boldsymbol{H}=\\&-(\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})^{*}\bar{\boldsymbol{Y}})^{\mathrm T}\mathrm{D}\boldsymbol{x}(\boldsymbol{0})\boldsymbol{H}=\\&-\langle\bar{\boldsymbol{Y}},\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\mathrm{D}\boldsymbol{x}(\boldsymbol{0})\boldsymbol{H}\rangle=\\&-\langle\boldsymbol{P}^{\mathrm T}\bar{\boldsymbol{Y}}\boldsymbol{P},\boldsymbol{P}^{\mathrm T}\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\mathrm{D}\boldsymbol{x}(\boldsymbol{0})\boldsymbol{H}\boldsymbol{P}^{\mathrm T}\rangle=\\&-\langle\boldsymbol{\Lambda}_{\alpha\alpha},(\boldsymbol{P}^{\mathrm T}\mathrm{D}\boldsymbol{G}(\bar{\boldsymbol{x}})\mathrm{D}\boldsymbol{x}(\boldsymbol{0})\boldsymbol{H}\boldsymbol{P}^{\mathrm T})_{\alpha\alpha}\rangle=\\&-\langle\boldsymbol{\Lambda}_{\alpha\alpha},(\boldsymbol{P}^{\mathrm T}\boldsymbol{H}\boldsymbol{P})_{\alpha\alpha}\rangle=\\&-\langle\boldsymbol{P}^{\mathrm T}\bar{\boldsymbol{Y}}\boldsymbol{P},\boldsymbol{P}^{\mathrm T}\boldsymbol{H}\boldsymbol{P}\rangle=\\&-\langle\bar{\boldsymbol{Y}},\boldsymbol{H}\rangle\end{aligned}$$

得到结论.

2　一类双层规划的最优性条件

考虑如下的双层优化问题,上层优化问题定义为

$$\begin{aligned}\min\quad&\theta(\boldsymbol{x},\boldsymbol{u})\\\text{s.t.}\quad&\boldsymbol{x}\in\mathrm{Sol}\ \mathrm{P}(\boldsymbol{u})\\&\boldsymbol{u}\in\boldsymbol{U}_{ad}\end{aligned} \tag{17}$$

下层为问题 P($\boldsymbol{u}$),定义如下:

$$\begin{aligned}\min\quad&f(\boldsymbol{x})\\\text{s.t.}\quad&\boldsymbol{G}(\boldsymbol{x})+\boldsymbol{B}\boldsymbol{u}\in\boldsymbol{S}_{-}^{p}\\&\boldsymbol{x}\in\boldsymbol{R}^{n}\end{aligned} \tag{18}$$

其中 $\theta:\boldsymbol{R}^n\times\boldsymbol{R}^m\to\mathbf{R}$ 是连续可微函数,$\boldsymbol{U}_{ad}\subset\boldsymbol{R}^m$ 是非空闭凸集合,Sol P($\boldsymbol{u}$) 表示问题 P($\boldsymbol{u}$) 的最优解集合,$\boldsymbol{B}:\boldsymbol{R}^m\to\boldsymbol{S}^p$ 是一连续的线性算子.

对任何 $\boldsymbol{u}\in\boldsymbol{R}^m$,设在 $(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{Y}(\boldsymbol{u}))$ 处问题 P($\boldsymbol{u}$) 的雅可比唯一性条件成立,由命题 1 得,$(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{Y}(\boldsymbol{u}))$ 是二次连续可微映射,满足

$$\begin{aligned}&\nabla_x L_0(\boldsymbol{x},\boldsymbol{u},\boldsymbol{Y})=\boldsymbol{0}\\&\Pi_{\boldsymbol{S}_{+}^{p}}(\boldsymbol{Y}+\boldsymbol{G}(\boldsymbol{x})+\boldsymbol{B}\boldsymbol{u})-\boldsymbol{Y}=\boldsymbol{0}\end{aligned} \tag{19}$$

其中

$$L_0(\boldsymbol{x},\boldsymbol{u},\boldsymbol{Y})=f(\boldsymbol{x})+\langle\boldsymbol{Y},\boldsymbol{G}(\boldsymbol{x})+\boldsymbol{B}\boldsymbol{u}\rangle$$

令 $\boldsymbol{V}(\boldsymbol{u})=\mathrm{D}\Pi_{\boldsymbol{S}_{+}^{p}}(\boldsymbol{Y}(\boldsymbol{u})+\boldsymbol{G}(\boldsymbol{x}(\boldsymbol{u}))+\boldsymbol{B}\boldsymbol{u}):\boldsymbol{R}^m\to\boldsymbol{S}^p$,可得到下述的最优性必要条件.

命题 2　设 $f:\boldsymbol{R}^n\to\mathbf{R}$ 与 $\boldsymbol{G}:\boldsymbol{R}^n\to\boldsymbol{S}_{-}^{p}$ 是二次连续可微函数和映射,对每一 $\boldsymbol{u}\in\boldsymbol{R}^m$,$(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{Y}(\boldsymbol{u}))$ 处问题 P($\boldsymbol{u}$) 的雅可比唯一性条件成立.如果 $\boldsymbol{u}^{*}\in\boldsymbol{U}_{ad}$ 是式(17)的局部极小点,则

$$\langle-\boldsymbol{B}^{*}(\boldsymbol{V}(\boldsymbol{u}^{*})\boldsymbol{P}_2)+\nabla_u\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*}),\boldsymbol{u}-\boldsymbol{u}^{*}\rangle\geqslant 0;\quad\forall\,\boldsymbol{u}\in\boldsymbol{U}_{ad} \tag{20}$$

其中 $(\boldsymbol{P}_1,\boldsymbol{P}_2)\in\boldsymbol{R}^n\times\boldsymbol{S}^p$ 满足如下的伴随方程:

$$\begin{pmatrix}\nabla_{xx}^2L(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{Y}(\boldsymbol{u}^{*}))&\mathrm{D}\boldsymbol{G}(\boldsymbol{x}(\boldsymbol{u}^{*}))^{*}\\\boldsymbol{V}(\boldsymbol{u}^{*})\mathrm{D}\boldsymbol{G}(\boldsymbol{x}(\boldsymbol{u}^{*}))&\boldsymbol{V}(\boldsymbol{u}^{*})-\boldsymbol{I}\end{pmatrix}^{*}\begin{pmatrix}\boldsymbol{P}_1\\\boldsymbol{P}_2\end{pmatrix}=\begin{pmatrix}\nabla_x\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*})\\\boldsymbol{0}\end{pmatrix} \tag{21}$$

证明　定义 $\theta_0(\boldsymbol{u})=\theta(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{u})$.如果 $\boldsymbol{u}^{*}\in\boldsymbol{U}_{ad}$ 是式(17)的局部极小点,则

$$\langle\nabla\theta_0(\boldsymbol{u}^{*}),\boldsymbol{u}-\boldsymbol{u}^{*}\rangle\geqslant 0;\quad\forall\,\boldsymbol{u}\in\boldsymbol{U}_{ad} \tag{22}$$

令

$$\boldsymbol{M}(\boldsymbol{u})=\begin{pmatrix}\nabla_{xx}^2L(\boldsymbol{x}(\boldsymbol{u}),\boldsymbol{Y}(\boldsymbol{u}))&\mathrm{D}\boldsymbol{G}(\boldsymbol{x}(\boldsymbol{u}))^{*}\\\boldsymbol{V}(\boldsymbol{u})\mathrm{D}\boldsymbol{G}(\boldsymbol{x}(\boldsymbol{u}))&\boldsymbol{V}(\boldsymbol{u})-\boldsymbol{I}\end{pmatrix}$$

则

$$\boldsymbol{M}(\boldsymbol{u}^{*})\begin{pmatrix}\boldsymbol{J}\boldsymbol{x}(\boldsymbol{u}^{*})(\boldsymbol{u}-\boldsymbol{u}^{*})\\\mathrm{D}\boldsymbol{Y}(\boldsymbol{u}^{*})(\boldsymbol{u}-\boldsymbol{u}^{*})\end{pmatrix}=\begin{pmatrix}\boldsymbol{0}\\-\boldsymbol{V}(\boldsymbol{u}^{*})\boldsymbol{B}(\boldsymbol{u}-\boldsymbol{u}^{*})\end{pmatrix}$$

注意到式(21)可以表示为

$$\boldsymbol{M}(\boldsymbol{u}^{*})^{*}\begin{pmatrix}\boldsymbol{P}_1\\\boldsymbol{P}_2\end{pmatrix}=\begin{pmatrix}\nabla_x\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*})\\\boldsymbol{0}\end{pmatrix}$$

得到对 $\boldsymbol{u}\in\boldsymbol{U}_{ad}$,

$$\begin{aligned}0\leqslant&\langle\nabla\theta_0(\boldsymbol{u}^{*}),\boldsymbol{u}-\boldsymbol{u}^{*}\rangle=\\&\langle\boldsymbol{J}\boldsymbol{x}(\boldsymbol{u}^{*})^{\mathrm T}\nabla_x\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*})+\\&\nabla_u\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*}),\boldsymbol{u}-\boldsymbol{u}^{*}\rangle=\\&\left\langle\begin{pmatrix}\nabla_x\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*})\\\boldsymbol{0}\end{pmatrix},\begin{pmatrix}\boldsymbol{J}\boldsymbol{x}(\boldsymbol{u}^{*})(\boldsymbol{u}-\boldsymbol{u}^{*})\\\mathrm{D}\boldsymbol{Y}(\boldsymbol{u}^{*})(\boldsymbol{u}-\boldsymbol{u}^{*})\end{pmatrix}\right\rangle+\\&\langle\nabla_u\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*}),\boldsymbol{u}-\boldsymbol{u}^{*}\rangle=\\&\left\langle[\boldsymbol{M}(\boldsymbol{u}^{*})^{-1}]^{*}\begin{pmatrix}\nabla_x\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*})\\\boldsymbol{0}\end{pmatrix},\right.\\&\left.\boldsymbol{M}(\boldsymbol{u}^{*})\begin{pmatrix}\boldsymbol{J}\boldsymbol{x}(\boldsymbol{u}^{*})(\boldsymbol{u}-\boldsymbol{u}^{*})\\\mathrm{D}\boldsymbol{Y}(\boldsymbol{u}^{*})(\boldsymbol{u}-\boldsymbol{u}^{*})\end{pmatrix}\right\rangle+\\&\langle\nabla_u\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*}),\boldsymbol{u}-\boldsymbol{u}^{*}\rangle=\\&\left\langle\begin{pmatrix}\boldsymbol{P}_1\\\boldsymbol{P}_2\end{pmatrix},\begin{pmatrix}\boldsymbol{0}\\-\boldsymbol{V}(\boldsymbol{u}^{*})\boldsymbol{B}(\boldsymbol{u}-\boldsymbol{u}^{*})\end{pmatrix}\right\rangle+\\&\langle\nabla_u\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*}),\boldsymbol{u}-\boldsymbol{u}^{*}\rangle=\\&\langle-\boldsymbol{B}^{*}(\boldsymbol{V}(\boldsymbol{u}^{*})\boldsymbol{P}_2)+\nabla_u\theta(\boldsymbol{x}(\boldsymbol{u}^{*}),\boldsymbol{u}^{*}),\\&\boldsymbol{u}-\boldsymbol{u}^{*}\rangle\end{aligned}$$

即式(20)成立.

3 结 语

本文证明了非线性半定规划的雅可比唯一性定理,扰动问题的函数是决策变量与扰动参数的二次连续可微函数时的扰动解的连续可微性质,扰动函数的导数,以及一类下层为非线性半定规划的特殊双层规划的最优性条件. 在雅可比唯一性条件中,严格互补条件是至关重要的,如果这一条件不成立,非线性半定规划的扰动性分析需要用到正半定矩阵锥的非光滑分析. 非线性系统的强正则性和映射的 Lipschtz 同胚,与约束非退化条件和强二阶充分性最优条件等详见文献[7].

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Jacobian uniqueness theorem for nonlinear semidefinite programming

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Abstract: Jacobian uniqueness conditions for nonlinear semidefinite programming are presented, under which, the mapping characterizing the KKT conditions is demonstrated to have a nonsingular derivative at the KKT point. Under the Jacobian uniqueness conditions, stability theorem for nonlinear semidefinite programming is proved and the necessary optimality conditions for a special bi-level program, whose lower level problem is a nonlinear semidefinite optimization problem, are established.

Key words: semidefinite programming; Jacobian uniqueness conditions; stability; bi-level program