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具有广义多面体约束的参数变分不等式解映射伴同导数

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摘要: 在研究参数变分不等式稳定性理论及均衡约束数学规划的最优性条件时,计算参数变分不等式解映射的伴同导数显得尤为重要.考虑了具有等式约束的广义多面体约束的参数不等式.首先,在无约束规范条件下,利用二阶微分理论,给出了具有广义多面体约束的法锥的图的法锥.其次,借助辅助多面体集合及约束规范条件,得到了更为简洁的法锥形式.最后,给出参数变分不等式的解映射的伴同导数.

关键词: 参数变分不等式;广义多面体;伴同导数

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0 引 言

在 Banach 空间 $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ 及其对偶空间 $\boldsymbol{X}^*$, $\boldsymbol{Y}^*$, $\boldsymbol{Z}^*$ 中,考虑下面的参数变分不等式:

$$\boldsymbol{v}\in f(\boldsymbol{x},\boldsymbol{p})+N(\boldsymbol{x};\boldsymbol{\Omega});\text{对所有的 }\boldsymbol{x}\in\boldsymbol{\Omega}$$

$\boldsymbol{x}\in\boldsymbol{X}$ 为决策变量,$\boldsymbol{v}$、$\boldsymbol{p}$ 为扰动参量,其中 $\boldsymbol{v}\in\boldsymbol{X}^*$ 是标准扰动变量,$\boldsymbol{p}\in\boldsymbol{Z}$ 是基本扰动变量.函数 f: $\boldsymbol{X}\times\boldsymbol{Z}\to\boldsymbol{X}^*$ 严格可微.上述变分不等式可以等价地表示为下面的标准变分不等式:对 $\boldsymbol{v}\in\boldsymbol{X}^*$, $\boldsymbol{p}\in\boldsymbol{Z}$,存在 $\boldsymbol{x}\in\boldsymbol{X}$ 使得

$$\langle f(\boldsymbol{x},\boldsymbol{p})-\boldsymbol{v},\boldsymbol{z}-\boldsymbol{x}\rangle\geqslant 0;\text{对所有的 }\boldsymbol{z}\in\boldsymbol{\Omega}$$

许多优化问题,比如参数互补问题、均衡问题、KKT 系统等都可以用上述参数变分不等式表示.在本文中考虑 $\boldsymbol{\Omega}$ 是下面定义的广义多面体:

$$\boldsymbol{\Omega}=\{\boldsymbol{x}\in\boldsymbol{X}\,|\,\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b},\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle\leqslant c_i,i\in T\}$$

其中 $\boldsymbol{A}:\boldsymbol{X}\to\boldsymbol{Y}$ 为线性有界算子,$\{\boldsymbol{x}_i^*\}$ 为一列线性有界算子,$T=\{1,2,\cdots,m\}(m\geqslant 1)$.记 $S:\boldsymbol{X}^*\times\boldsymbol{Z}\Rightarrow\boldsymbol{X}$ 为参数变分不等式的解映射,即

$$S(\boldsymbol{v},\boldsymbol{p})=\{\boldsymbol{x}\in\boldsymbol{X}\,|\,\boldsymbol{v}\in f(\boldsymbol{x},\boldsymbol{p})+N(\boldsymbol{x};\boldsymbol{\Omega})\}$$

近年来许多学者对参数变分不等式进行了研究,而且取得了不少成果[1-3].而参数变分不等式的解映射也是研究的一个极其重要的方面,它为具有均衡约束的优化问题的最优性条件[4]及稳定性分析[1]提供了重要的理论依据.近年来,对这一问题研究取得了许多重要的进展.Henrion 等在文献[5]中利用二阶微分理论得到了无限维 Banach空间中具有多面体约束的参数变分不等式解映射的伴同导数.此后,Ban 等在文献[6]中建立了广义多面体约束的参数变分不等式的解映射的伴同导数.此外,以 c 为变量得到扰动集合 $\boldsymbol{\Omega}(c)$,文献[7-8]建立了此类多面体约束的参数变分不等式的微分理论.本文将文献[5]中的多面体推广到具有线性等式约束的广义多面体,在半无限线性规划及无限维空间的广义线性规划中,许多问题的约束都可以写成上述形式的广义多面体,因此研究广义多面体集合具有十分重要的意义.对于带有线性等式约束的广义多面体,为了得到广义多面体集合法锥映射的图的极限法锥的简洁形式,不能像文献[5]中定理 4.2 那样直接证明得到,因此借助定义的辅助多面体集合,首先建立辅助多面体集合的法锥映射的图的极限法锥的简

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洁形式，然后建立广义多面体的法锥映射的图的极限法锥的简洁形式，最后建立具有广义多面体约束的参数变分不等式解映射的伴同导数.

1　预备知识

在这一部分中，介绍一些变分分析[9]中的基本概念. 首先介绍 Frechet 法锥. 在非空集合 $\boldsymbol{\Gamma}\subseteq\boldsymbol{X}$ 中，令 $\bar{\boldsymbol{x}}\in\boldsymbol{\Gamma}$，定义

$$\widehat{N}(\bar{\boldsymbol{x}};\boldsymbol{\Gamma})=\left\{\boldsymbol{x}^*\in\boldsymbol{X}^*\ \middle|\ \limsup_{\boldsymbol{x}\to\bar{\boldsymbol{x}}}\frac{\langle\boldsymbol{x}^*,\boldsymbol{x}-\bar{\boldsymbol{x}}\rangle}{\|\boldsymbol{x}-\bar{\boldsymbol{x}}\|}\leqslant 0\right\}\tag{1}$$

为 $\boldsymbol{\Gamma}$ 在 $\bar{\boldsymbol{x}}$ 处的 Frechet 法锥（又叫预法锥或正则法锥）. 在 $\bar{\boldsymbol{x}}\in\boldsymbol{\Gamma}$ 处的基本法锥为

$$N(\bar{\boldsymbol{x}};\boldsymbol{\Gamma})=\limsup_{\boldsymbol{x}\to\bar{\boldsymbol{x}}}\widehat{N}(\boldsymbol{x};\boldsymbol{\Gamma})\tag{2}$$

（又叫极限法锥或者 Mordukhovich 法锥）. 当 $\boldsymbol{\Gamma}$ 为凸集时，则上述定义的 Frechet 法锥和极限法锥与凸分析中定义的法锥一致. 设 $F:\boldsymbol{X}\rightrightarrows\boldsymbol{Y}$ 是一闭图的集值映射，$(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})\in\operatorname{gph}F:=\{(\boldsymbol{x},\boldsymbol{y})\in\boldsymbol{X}\times\boldsymbol{Y}\mid\boldsymbol{y}\in F(\boldsymbol{x})\}$，$F$ 在点 $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$ 的 Frechet 伴同导数（又叫预伴同导数）$\widehat{D}^*F(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}):\boldsymbol{Y}^*\rightrightarrows\boldsymbol{X}^*$ 定义为

$$\widehat{D}^*F(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})(\boldsymbol{y}^*)=\{\boldsymbol{x}^*\in\boldsymbol{X}^*\mid(\boldsymbol{x}^*,-\boldsymbol{y}^*)\in\widehat{N}((\bar{\boldsymbol{x}},\bar{\boldsymbol{y}});\operatorname{gph}F)\}\tag{3}$$

伴同导数（或极限伴同导数或 Mordukhovich 伴同导数）定义为

$$D^*F(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})(\boldsymbol{y}^*)=\{\boldsymbol{x}^*\in\boldsymbol{X}^*\mid(\boldsymbol{x}^*,-\boldsymbol{y}^*)\in N((\bar{\boldsymbol{x}},\bar{\boldsymbol{y}});\operatorname{gph}F)\}\tag{4}$$

由 $\{\boldsymbol{v}_j^*\mid j\in J\}$ 产生的子空间的核表示为 $\ker\{\boldsymbol{v}_j^*\mid j\in J\}=\{\boldsymbol{x}\in\boldsymbol{X}\mid\langle\boldsymbol{v}_j^*,\boldsymbol{x}\rangle=0,j\in J\}$，这与 $\ker\boldsymbol{A}=\{\boldsymbol{x}\in\boldsymbol{X}\mid\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\}$ 一致，记 $\{\boldsymbol{v}^*\}^{\perp}=\{\boldsymbol{x}\in\boldsymbol{X}\mid\langle\boldsymbol{v}^*,\boldsymbol{x}\rangle=0\}$. 下面给出择一性定理，参见文献[5].

定理 1　$\boldsymbol{W}$ 为向量空间，$\boldsymbol{A}:\boldsymbol{W}\to\boldsymbol{R}^d$ 和 $\boldsymbol{B}:\boldsymbol{W}\to\boldsymbol{R}^s$ 为线性映射. 则下述两系统之一有解：

（1）$\exists\,\boldsymbol{x}\in\boldsymbol{W}$ 使得 $\boldsymbol{B}\boldsymbol{x}\geqslant\boldsymbol{0},\boldsymbol{A}\boldsymbol{x}>\boldsymbol{0}$；

（2）$\exists\,\boldsymbol{\lambda}\in\boldsymbol{R}^d$，$\exists\,\boldsymbol{\mu}\in\boldsymbol{R}^s$ 使得 $\boldsymbol{\lambda}\geqslant\boldsymbol{0},\boldsymbol{\lambda}\neq\boldsymbol{0},\boldsymbol{\mu}\geqslant\boldsymbol{0}$，$\sum_{i=1}^{d}\lambda_iA_i+\sum_{j=1}^{s}\lambda_jB_j=0$，其中 A_i、B_j 分别为 $\boldsymbol{A}$、$\boldsymbol{B}$ 的分量.

2　广义多面体约束的法锥映射的伴同导数

定义法锥映射 $\mathscr{F}:\boldsymbol{X}\rightrightarrows\boldsymbol{X}^*$ 为 $\mathscr{F}(\boldsymbol{x}):=N(\boldsymbol{x};\boldsymbol{\Omega})$，$\boldsymbol{x}\in\boldsymbol{X}$，及有效约束指标集为 $I(\bar{\boldsymbol{x}}):=\{i\in T\mid\langle\boldsymbol{x}_i^*,\bar{\boldsymbol{x}}\rangle=c_i\}$，则集合 $\boldsymbol{\Omega}$ 法锥为

$$N(\bar{\boldsymbol{x}};\boldsymbol{\Omega})=\boldsymbol{A}^*\boldsymbol{Y}^*+\left\{\sum_{i\in I(\bar{x})}\lambda_ix_i^*\ \middle|\ \lambda_i\geqslant 0\right\}\tag{5}$$

对 $\bar{\boldsymbol{x}}^*\in\mathscr{F}(\bar{\boldsymbol{x}})$，由式(5)得

$$\bar{\boldsymbol{x}}^*=A^*\bar{\boldsymbol{y}}^*+\sum_{i\in I(\bar{\boldsymbol{x}}),\lambda_i\geqslant 0}\lambda_i\boldsymbol{x}_i^*;\ \bar{\boldsymbol{y}}^*\in\boldsymbol{Y}^*$$

记 $J(\boldsymbol{\lambda}):=\{i\in I(\bar{\boldsymbol{x}})\mid\lambda_i>0\}$ 为所有大于零的乘子的集合. 给定任意的 $P\subseteq Q\subseteq T$，定义

$$\mathscr{A}_{Q,P}:=\operatorname{span}\{\boldsymbol{x}_i^*\mid i\in P\}+\operatorname{pos}\{\boldsymbol{x}_i^*\mid i\in Q\backslash P\},$$

$$\mathscr{B}_{Q,P}:=\{\boldsymbol{x}\in\boldsymbol{X}\mid\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle=0,i\in P,\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle\leqslant 0,\ i\in Q\backslash P\}$$

其中

$$\operatorname{span}\{\boldsymbol{x}_i^*\mid i\in P\}=\left\{\sum_{i\in P}\lambda_i\boldsymbol{x}_i^*\ \middle|\ \lambda_i\in\mathbf{R},i\in P\right\}$$

$$\operatorname{pos}\{\boldsymbol{x}_i^*\mid i\in Q\backslash P\}=\operatorname{cone}\{\boldsymbol{x}_i^*\mid i\in Q\backslash P\}$$

一般情况下，指标集 Q 取为指标集 $I(\bar{\boldsymbol{x}})$ 的子集，指标集 P 取为指标集 $J(\boldsymbol{\lambda})$ 的子集. 下面给出在无约束规范条件下，映射 $\mathscr{F}$ 的图的 Frechet 法锥及极限法锥.

定理 2[6]　设 $(\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*)\in\operatorname{gph}\mathscr{F}$，$I:=I(\bar{\boldsymbol{x}})$，$J:=J(\boldsymbol{\lambda})$，$\mathscr{A}_{I,J}$ 和 $\mathscr{B}_{I,J}$ 如上面定义. 则

$$\widehat{N}((\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*);\operatorname{gph}\mathscr{F})=[\mathscr{A}_{I,J}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{I,J}\cap\ker\boldsymbol{A}]$$

固定 $Q\subseteq T$，令

$$\mathscr{S}_Q:=\{\boldsymbol{x}\in\boldsymbol{X}\mid\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle=c_i,i\in Q;\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle<c_i,i\in T\backslash Q,\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}\}$$

由式(5)知，集合 $\boldsymbol{\Omega}$ 的法锥又可以表示为 $N(\bar{\boldsymbol{x}};\boldsymbol{\Omega})=\boldsymbol{A}^*\boldsymbol{Y}^*+\operatorname{pos}\{\boldsymbol{x}_i^*\mid i\in I(\bar{\boldsymbol{x}})\}$. 由于 $P\subset I(\bar{\boldsymbol{x}})$，记指标集 $\mathscr{I}(\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*):=\{P\subset I(\bar{\boldsymbol{x}})\mid\bar{\boldsymbol{x}}^*\in\boldsymbol{A}^*\boldsymbol{Y}^*+\operatorname{pos}\{\boldsymbol{x}_i^*\mid i\in P\}\}$ 为由 P 和 $N(\bar{\boldsymbol{x}};\boldsymbol{\Omega})$ 生成的指标集集合.

定理 3　给定 $(\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*)\in\operatorname{gph}\mathscr{F}$，$\mathscr{I}:=\mathscr{I}(\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*)$，$\mathscr{S}_Q$ 如上述定义. 则

$$N((\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*);\operatorname{gph}\mathscr{F})=\bigcup_{P\subset Q\subset I,P\in\mathscr{I},\mathscr{S}_Q\neq\varnothing}[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times$$

$$[\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}] \tag{6}$$

证明 首先证明"$\subseteq$". 取$(\boldsymbol{v}^*,\boldsymbol{u})\in N((\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*);\mathrm{gph}\,\mathscr{F})$,由法锥的定义式(2)知,存在$\mathbf{N}$使得$k\in\mathbf{N}$,$(\boldsymbol{v}_k^*,\boldsymbol{u}_k)\in\hat{N}((\boldsymbol{x}_k,\boldsymbol{z}_k^*);\mathrm{gph}\,\mathscr{F})$,且$k\to\infty$,$(\boldsymbol{x}_k,\boldsymbol{z}_k^*)\xrightarrow{\mathrm{gph}\,\mathscr{F}}(\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*)$,$(\boldsymbol{v}_k^*,\boldsymbol{u}_k)\xrightarrow{w\times w}(\boldsymbol{v}^*,\boldsymbol{u})$. 则$\boldsymbol{x}_k\in\boldsymbol{\Omega}$,$\boldsymbol{z}_k^*\in N(\boldsymbol{x}_k;\boldsymbol{\Omega})$. 令$I(\boldsymbol{x}_k)=\{i\in T\mid\langle\boldsymbol{x}_i^*,\boldsymbol{x}_k\rangle=c_i\}$为有效约束指标集,则有$I(\boldsymbol{x}_k)\subset I(\bar{\boldsymbol{x}})$. 取$Q=I(\boldsymbol{x}_k)$,则$Q\subset I$,$\boldsymbol{x}_k\in\mathscr{S}_Q$,即$\mathscr{S}_Q\neq\varnothing$. 又$\boldsymbol{z}_k^*\in N(\boldsymbol{x}_k;\boldsymbol{\Omega})$,$Q=I(\boldsymbol{x}_k)$,则$\boldsymbol{z}_k^*=\boldsymbol{A}^*\boldsymbol{y}_k^*+\sum\limits_{i\in Q}\lambda_{ik}\boldsymbol{x}_i^*$,$\boldsymbol{y}_k^*\in\boldsymbol{Y}$,$\lambda_{ik}\geqslant0$,$k\in\mathbf{N}$. 取$P\subseteq Q\subseteq I$满足$P:=\{i\in Q\mid\lambda_{ik}>0\}$,$k\in\mathbf{N}$. 则$\boldsymbol{z}_k^*=\boldsymbol{A}^*\boldsymbol{y}_k^*+\sum\limits_{i\in P}\lambda_{ik}\boldsymbol{x}_i^*\in\boldsymbol{A}^*\boldsymbol{Y}^*+\mathrm{pos}\{\boldsymbol{x}_i^*\mid i\in P\}$,即$P\in\mathscr{I}$. 由定理2知:$(\boldsymbol{v}_k^*,\boldsymbol{u}_k)\in[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}]$. 令$k\to\infty$,则$(\boldsymbol{v}^*,\boldsymbol{u})\in[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}]$.

证明"$\supseteq$". 取$(\boldsymbol{v}^*,\boldsymbol{u})\in\bigcup\limits_{P\subset Q\subset I,P\in\mathscr{I},\mathscr{S}_Q\neq\varnothing}[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}]$,则存在$P\subseteq Q\subseteq I$满足$P\in\mathscr{I}$,$(\boldsymbol{v}^*,\boldsymbol{u})\in[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}]$,$\mathscr{S}_Q\neq\varnothing$. 固定$\hat{\boldsymbol{x}}\in\mathscr{S}_Q$,则$\langle\boldsymbol{x}_i^*,\hat{\boldsymbol{x}}\rangle=c_i$,$i\in Q$;$\langle\boldsymbol{x}_i^*,\hat{\boldsymbol{x}}\rangle<c_i$,$i\in T\setminus Q$;$\boldsymbol{A}\hat{\boldsymbol{x}}=\boldsymbol{b}$. 构造$\{\boldsymbol{x}_k\}\subset\boldsymbol{X}$,$\boldsymbol{x}_k=\frac{1}{k}\hat{\boldsymbol{x}}+\left(1-\frac{1}{k}\right)\bar{\boldsymbol{x}}$,$k\in\mathbf{N}$,则$\boldsymbol{x}_k\to\bar{\boldsymbol{x}}$,$k\to\infty$,$\boldsymbol{x}_k\in\mathscr{S}_Q$,即$\boldsymbol{x}_k\in\boldsymbol{\Omega}$. 在$\boldsymbol{x}_k$处,$I(\boldsymbol{x}_k)$即为$Q$,则$N(\boldsymbol{x}_k;\boldsymbol{\Omega})=\boldsymbol{A}^*\boldsymbol{Y}^*+\mathrm{pos}\{\boldsymbol{x}_i^*\mid i\in Q\}$,$k\in\mathbf{N}$. 又$P\in\mathscr{I}$,则$\bar{\boldsymbol{x}}^*=\boldsymbol{A}^*\boldsymbol{y}^*+\sum\limits_{i\in P}\lambda_i\boldsymbol{x}_i^*$,$\boldsymbol{y}^*\in\boldsymbol{Y}^*$,$\lambda_i\geqslant0$. 定义$\{\boldsymbol{z}_k^*\}\subset\boldsymbol{X}^*$,$\boldsymbol{z}_k^*=\boldsymbol{A}^*\boldsymbol{y}_k^*+\sum\limits_{i\in P}\left(\lambda_i+\frac{1}{k}\right)\boldsymbol{x}_i^*$,$\|\boldsymbol{z}_k^*-\bar{\boldsymbol{x}}^*\|\to0$,$k\to\infty$. 而$P\subseteq Q$,则$\boldsymbol{z}_k^*\in N(\boldsymbol{x}_k;\boldsymbol{\Omega})$. 将$(\boldsymbol{x}_k,\boldsymbol{z}_k^*)$应用到定理2知:$(\boldsymbol{v}^*,\boldsymbol{u})\in[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}]=\hat{N}((\boldsymbol{x}_k,\boldsymbol{z}_k^*);\mathrm{gph}\,\mathscr{F})$. 令$k\to\infty$,则$(\boldsymbol{v}^*,\boldsymbol{u})\in N((\boldsymbol{x}_k,\boldsymbol{z}_k^*);\mathrm{gph}\,\mathscr{F})$,所以结论成立.

为了得到$N((\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*);\mathrm{gph}\,\mathscr{F})$的简洁形式,首先要借助辅助多面体集合的法锥的简洁形式. 下面给出辅助多面体集合:

$$\boldsymbol{\Theta}=\{\boldsymbol{x}\in\boldsymbol{X}\mid\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle\leqslant0,i\in T,\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\} \tag{7}$$

固定$\tilde{\boldsymbol{x}}\in\boldsymbol{\Theta}$,在$\tilde{\boldsymbol{x}}$处有效约束指标集为$\bar{I}(\tilde{\boldsymbol{x}})=\{i\in T\mid\langle\boldsymbol{x}_i^*,\tilde{\boldsymbol{x}}\rangle=0\}$. 选取$\bar{Q}\subseteq T$,则记$\tilde{\mathscr{S}}_{\bar{Q}}:=\{\boldsymbol{x}\in\boldsymbol{X}\mid\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle=0,i\in\bar{Q};\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle<0,i\in T\setminus\bar{Q},\boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\}$,$\mathscr{G}(\boldsymbol{x}):=N(\boldsymbol{x};\boldsymbol{\Theta})=\boldsymbol{A}^*\boldsymbol{Y}^*+\left\{\sum\limits_{i\in\bar{I}(\tilde{\boldsymbol{x}})}\mu_i\boldsymbol{x}_i^*\mid\mu_i\geqslant0\right\}$,$\bar{J}(\boldsymbol{\mu}):=\{i\in\bar{I}(\tilde{\boldsymbol{x}})\mid\mu_i>0\}$,$\bar{\mathscr{I}}(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}^*):=\{P\subset I(\tilde{\boldsymbol{x}})\mid\tilde{\boldsymbol{x}}^*\in\boldsymbol{A}^*\boldsymbol{Y}^*+\mathrm{pos}\{\boldsymbol{x}_i^*\mid i\in P\}\}$. 为证明方便,简记$\bar{I}:=\bar{I}(\tilde{\boldsymbol{x}})$,$\bar{\mathscr{I}}:=\bar{\mathscr{I}}(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}^*)$,$\bar{J}:=\bar{J}(\boldsymbol{\mu})$,下面给出$N((\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}^*);\mathrm{gph}\,\mathscr{G})$的更一般的形式.

定理4 令$(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}^*)\in\mathrm{gph}\,\mathscr{G}$,假设$\{\boldsymbol{x}_i^*\mid i\in\bar{I}\}$线性无关及"约束规范条件"$(\ker\boldsymbol{A})^{\perp}\cap\mathrm{span}\{\boldsymbol{x}_i^*\mid i\in\bar{I}(\tilde{\boldsymbol{x}})\}=\{0\}$成立,则

$$N((\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}^*);\mathrm{gph}\,\mathscr{G})=\bigcup_{\bar{J}\subset\bar{P}\subset\bar{Q}\subset\bar{I}}[\mathscr{A}_{\bar{Q},\bar{P}}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{\bar{Q},\bar{P}}\cap\ker\boldsymbol{A}] \tag{8}$$

证明 将定理3应用到辅助多面体集合得

$$N((\tilde{\boldsymbol{x}},\tilde{\boldsymbol{x}}^*);\mathrm{gph}\,\mathscr{G})=\bigcup_{\bar{P}\subset\bar{Q}\subset\bar{I},\bar{P}\in\bar{\mathscr{I}},\tilde{\mathscr{S}}_{\bar{Q}}\neq\varnothing}[\mathscr{A}_{\bar{Q},\bar{P}}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{\bar{Q},\bar{P}}\cap\ker\boldsymbol{A}] \tag{9}$$

只需证明在假设的两个约束规范之下,式(9)可简化为式(8)即可. 首先证明:当$\bar{Q}\subseteq\bar{I}$时,$\tilde{\mathscr{S}}_{\bar{Q}}\neq\varnothing$. 当$\bar{Q}=\bar{I}$时,$\tilde{\boldsymbol{x}}\in\tilde{\mathscr{S}}_{\bar{I}}$,则$\tilde{\mathscr{S}}_{\bar{Q}}\neq\varnothing$成立. 否则,假设$\tilde{\mathscr{S}}_{\bar{Q}}\neq\varnothing$. 由择一性定理知,存在$\alpha_i$,$i\in\bar{Q}$,$\beta_j$,$j\in T\setminus\bar{Q}$,$\beta_j$至少有一个不为0,$\boldsymbol{y}^*\in\boldsymbol{Y}^*$,满足$\sum\limits_{i\in\bar{Q}}\alpha_i\boldsymbol{x}_i^*+\sum\limits_{j\in T\setminus\bar{Q}}\beta_j\boldsymbol{x}_j^*+\boldsymbol{A}^*\boldsymbol{y}^*=\boldsymbol{0}$. 由$\bar{I}(\tilde{\boldsymbol{x}})$的定义及$\bar{Q}\subset\bar{I}(\tilde{\boldsymbol{x}})$,有$\sum\limits_{j\in T\setminus\bar{I}}\beta_j\langle\boldsymbol{x}_j^*,\tilde{\boldsymbol{x}}\rangle=0$. 又$\langle\boldsymbol{x}_j^*,\tilde{\boldsymbol{x}}\rangle<0$,$j\in T\setminus\bar{I}$,则$\beta_j=0$,$j\in T\setminus\bar{I}$,则$\sum\limits_{i\in\bar{Q}}\alpha_i\boldsymbol{x}_i^*+\sum\limits_{j\in\bar{I}\setminus\bar{Q}}\beta_j\boldsymbol{x}_j^*+\boldsymbol{A}^*\boldsymbol{y}^*=\boldsymbol{0}$. 而$\boldsymbol{A}^*\boldsymbol{Y}^*=\ker\boldsymbol{A}$,则$\boldsymbol{\xi}=\sum\limits_{i\in\bar{Q}}\alpha_i\boldsymbol{x}_i^*+\sum\limits_{j\in\bar{I}\setminus\bar{Q}}\beta_j\boldsymbol{x}_j^*=-\boldsymbol{A}^*\boldsymbol{y}^*\in(\ker\boldsymbol{A})^{\perp}\cap\mathrm{span}\{\boldsymbol{x}_i^*\mid i\in\bar{I}\}$. 由约束规范条件成立知$\boldsymbol{\xi}=\sum\limits_{i\in\bar{Q}}\alpha_i\boldsymbol{x}_i^*+\sum\limits_{j\in\bar{I}\setminus\bar{Q}}\beta_j\boldsymbol{x}_j^*=\boldsymbol{0}$. 而$\{\boldsymbol{x}_i^*\mid i\in I\}$线性无关,则$\alpha_i=\beta_j=0$,这与$\beta_j$至少有一个不为0相矛盾,所以结论成立.

下面只需证明:$\bar{P}\in\bar{\mathscr{I}}\Leftrightarrow\bar{J}\subset\bar{P}$. 首先证明"$\Leftarrow$". 因为$\tilde{\boldsymbol{x}}^*\in\boldsymbol{A}^*\boldsymbol{Y}^*+\mathrm{pos}\{\boldsymbol{x}_i^*\mid i\in\bar{J}\}\subset\boldsymbol{A}^*\boldsymbol{Y}^*+\mathrm{pos}\{\boldsymbol{x}_i^*\mid i\in\bar{P}\}$,由$\bar{\mathscr{I}}$的定义知,$\bar{P}\in\bar{\mathscr{I}}$. 下面证明

“⇒”. 固定$\bar{P}\in\bar{\mathscr{I}}$，取 $\nu_i\geqslant 0, i\in\bar{P}, \boldsymbol{w}^*\in\boldsymbol{Y}^*$ 满足$\tilde{\boldsymbol{x}}^*=\boldsymbol{A}^*\boldsymbol{w}^*+\sum_{i\in\bar{P}}\nu_i\boldsymbol{x}_i^*$. 由$\bar{P}\subset\bar{I}$，及$\hat{N}(\tilde{\boldsymbol{x}},\boldsymbol{\theta})$的形式，令

$$\theta_i=\begin{cases}\mu_i; & i\in\bar{J}\\ 0; & i\in\bar{I}\backslash\bar{J}\end{cases}\qquad \eta_i=\begin{cases}\nu_i; & i\in\bar{P}\\ 0; & i\in\bar{I}\backslash\bar{P}\end{cases}\tag{10}$$

则存在$\tilde{\boldsymbol{w}}_1^*, \tilde{\boldsymbol{w}}_2^*\in\boldsymbol{Y}^*$ 满足$\tilde{\boldsymbol{x}}^*=\boldsymbol{A}^*\tilde{\boldsymbol{w}}_1^*+\sum_{i\in\bar{I}}\theta_i\boldsymbol{x}_i^*=\boldsymbol{A}^*\tilde{\boldsymbol{w}}_2^*+\sum_{i\in\bar{I}}\eta_i\boldsymbol{x}_i^*$，由此得到 $\sum_{i\in\bar{I}}(\theta_i-\eta_i)\boldsymbol{x}_i^*+\boldsymbol{A}^*(\tilde{\boldsymbol{w}}_1^*-\tilde{\boldsymbol{w}}_2^*)=\boldsymbol{0}$. 又 $\boldsymbol{A}^*\boldsymbol{Y}^*=\ker\boldsymbol{A}$，“约束规范条件”成立，则 $\boldsymbol{\rho}^*=\sum_{i\in\bar{I}}(\theta_i-\eta_i)\boldsymbol{x}_i^*=-\boldsymbol{A}^*(\tilde{\boldsymbol{w}}_1^*-\tilde{\boldsymbol{w}}_2^*)=\boldsymbol{0}$，而$\{\boldsymbol{x}_i^*\mid i\in\bar{I}\}$线性无关，则 $\theta_i-\eta_i=0$，即$\theta_i=\eta_i, i\in\bar{I}$. 假设$\bar{J}\subseteq\bar{P}$，则存在 $i\in\bar{I}$使得$i\in\bar{P}\backslash\bar{J}$. 由式(10)知 $0<\mu_i=\theta_i-\eta_i=0$，矛盾，所以$\bar{J}\subseteq\bar{P}$. 综上所述，结论成立.

定理 5　设$(\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*)\in\operatorname{gph}\mathscr{F}$，$\{\boldsymbol{x}_i^*\mid i\in I\}$线性无关，约束规范条件$(\ker\boldsymbol{A})^{\perp}\cap\operatorname{span}\{\boldsymbol{x}_i^*\mid i\in I(\bar{\boldsymbol{x}})\}=\{0\}$成立. 则

$$N((\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^*);\operatorname{gph}\mathscr{F})=\bigcup_{J\subseteq P\subseteq Q\subseteq I}[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}]\tag{11}$$

证明　由定理 4 知：首先需要验证 $\mathscr{S}_Q\neq\varnothing, Q\subseteq I$. 取$\tilde{\boldsymbol{x}}\in\boldsymbol{\Theta}$满足$\langle\boldsymbol{x}_i^*,\tilde{\boldsymbol{x}}\rangle=0, i\in I, \langle\boldsymbol{x}_i^*,\tilde{\boldsymbol{x}}\rangle<0, i\in T\backslash I, \boldsymbol{A}\tilde{\boldsymbol{x}}=\boldsymbol{0}$，即 $I=\bar{I}$. 固定$\bar{Q}=Q\subset T$，则$\tilde{\mathscr{S}}_{\bar{Q}}$变为$\tilde{\mathscr{S}}_Q=\{\boldsymbol{x}\in\boldsymbol{X}\mid\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle=0, i\in Q;\langle\boldsymbol{x}_i^*,\boldsymbol{x}\rangle<0, i\in T\backslash Q, \boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\}$，由定理 4 知，$\tilde{\mathscr{S}}_Q\neq\varnothing, Q\subset I$. 取$\tilde{\boldsymbol{x}}\in\tilde{\mathscr{S}}_Q$，则$\bar{\boldsymbol{x}}+\tilde{\boldsymbol{x}}\in\mathscr{S}_Q$，且 $\mathscr{S}_Q\neq\varnothing, Q\subset I$. 在约束规范条件成立及$\{\boldsymbol{x}_i^*\mid i\in I\}$线性无关的条件下，同定理 4 中的证明一样，可得 $P\in\mathscr{I}\Leftrightarrow J\subset P$. 综上所述，结论成立.

最后，给出具有广义多面体约束的参数变分不等式的解映射的伴同导数.

定理 6　假设$(\bar{\boldsymbol{v}},\bar{\boldsymbol{p}},\bar{\boldsymbol{x}})\in\operatorname{gph}S$. f 在$(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})$处严格可微，$\nabla_p f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})$非奇异. 令$\bar{\boldsymbol{x}}^*:=-f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})$，则下面结论成立：

(i)伴同导数 $D^*S(\bar{\boldsymbol{v}},\bar{\boldsymbol{p}},\bar{\boldsymbol{x}}):\boldsymbol{X}^*\rightrightarrows\boldsymbol{X}\times\boldsymbol{Z}^*$ 由下面所示计算可得：

$$\begin{aligned}D^*S(\bar{\boldsymbol{v}},\bar{\boldsymbol{p}},\bar{\boldsymbol{x}})(\boldsymbol{x}^*)=\{(\boldsymbol{p}^*,\boldsymbol{q}^*)\in\boldsymbol{X}\times\boldsymbol{Z}^*\mid\exists P\subset{}&Q\subset I, P\in\mathscr{I}, \mathscr{S}_Q\neq\varnothing\text{使 }\boldsymbol{q}^*=\\&-\nabla_p f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})^*\boldsymbol{p}^*,\\&(\nabla_x f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})^*\boldsymbol{p}^*-\boldsymbol{x}^*,\boldsymbol{p}^*)\\&\in[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times[\mathscr{B}_{Q,P}\cap\\&\ker\boldsymbol{A}]\}\end{aligned}$$

(ii)假设约束规范条件成立，$\{\boldsymbol{x}_i^*\mid i\in I\}$线性无关，则

$$\begin{aligned}D^*S(\bar{\boldsymbol{v}},\bar{\boldsymbol{p}},\bar{\boldsymbol{x}})(\boldsymbol{x}^*)=\{(\boldsymbol{p}^*,\boldsymbol{q}^*)\in\boldsymbol{X}\times\boldsymbol{Z}^*\mid\exists J\subset{}&P\subset Q\subset I\text{ 使 }\boldsymbol{q}^*=-\nabla_p f(\bar{\boldsymbol{x}},\\&\bar{\boldsymbol{p}})^*\boldsymbol{p}^*,(\nabla_x f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})^*\boldsymbol{p}^*-\\&\boldsymbol{x}^*,\boldsymbol{p}^*)\in[\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*]\times\\&[\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}]\}\end{aligned}$$

证明　(i)对$(\boldsymbol{v},\boldsymbol{p},\boldsymbol{x})\in\operatorname{gph}S$，定义映射 g：$\boldsymbol{X}^*\times\boldsymbol{Z}\times\boldsymbol{X}\to\boldsymbol{X}\times\boldsymbol{X}^*$ 为 $g(\boldsymbol{v},\boldsymbol{p},\boldsymbol{x}):=(\boldsymbol{x},\boldsymbol{v}-f(\boldsymbol{x},\boldsymbol{p}))$，则 $\operatorname{gph}S=\{(\boldsymbol{v},\boldsymbol{p},\boldsymbol{x})\in\boldsymbol{X}^*\times\boldsymbol{Z}\times\boldsymbol{X}\mid g(\boldsymbol{v},\boldsymbol{x},\boldsymbol{p})\in\operatorname{gph}\mathscr{F}\}=g^{-1}(\operatorname{gph}\mathscr{F})$. 应用文献[9]中定理 1.17，可得 $N((\bar{\boldsymbol{v}},\bar{\boldsymbol{p}},\bar{\boldsymbol{x}});\operatorname{gph}S)=\nabla g(\bar{\boldsymbol{v}},\bar{\boldsymbol{p}},\bar{\boldsymbol{x}})^*\times N((\bar{\boldsymbol{x}},\bar{\boldsymbol{v}}-f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}}));\operatorname{gph}\mathscr{F})$，由伴同导数的定义及定理 3 知存在 $P\subset Q\subset I, P\in\mathscr{I}, \mathscr{S}_Q\neq\varnothing$，满足

$$(\boldsymbol{p}^*,\boldsymbol{q}^*,-\boldsymbol{x}^*)\in\nabla g(\bar{\boldsymbol{v}},\bar{\boldsymbol{p}},\bar{\boldsymbol{x}})^*\begin{pmatrix}\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*\\ \mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}\end{pmatrix}$$

则存在 $\boldsymbol{u}\in\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*, \boldsymbol{u}^*\in\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}$ 满足 $\boldsymbol{p}^*=-\boldsymbol{u}, \boldsymbol{q}^*=\nabla_p f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})^*\boldsymbol{u}, \boldsymbol{x}^*=-\nabla_x f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})^*\boldsymbol{u}$，即对 $P\subseteq Q\subseteq I, P\in\mathscr{I}, \mathscr{S}_Q\neq\varnothing$，有 $\boldsymbol{q}^*=-\nabla_p f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})^*\boldsymbol{p}^*, \nabla_x f(\bar{\boldsymbol{x}},\bar{\boldsymbol{p}})^*\boldsymbol{p}^*-\boldsymbol{x}^*\in\mathscr{A}_{Q,P}+\boldsymbol{A}^*\boldsymbol{Y}^*$，$\boldsymbol{p}^*\in\mathscr{B}_{Q,P}\cap\ker\boldsymbol{A}$，则(i)的结论成立.

(ii)由定理 5 知，在两个约束规范条件之下，有式(11)成立，则(ii)同样成立.

3　结　论

本文主要建立了带有广义多面体约束的参数变分不等式的解映射的伴同导数. 从不同的角度对文献[6]进行了深入的推广. 首先，在无约束规范的条件下，得到了法锥映射的图的法锥. 其次，借助辅助多面体集合及约束规范，得到了更为一般的法锥形式. 最后，得到参数变分不等式的解映

射的伴同导数.

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Coderivative of solution mapping to parametric variational inequality constrained by generalized polyhedra

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Abstract: It is of great importance to compute the coderivative of the solution mapping to the parametric variational inequality while investigating the stability theory of the parametric variational inequality and optimality conditions of the mathematical programming governed by equilibrium constraints. A class of parametric variational inequalities constrained by generalized polyhedra including the equality constraints is considered. Firstly, by virtue of the second-order differentiation theory, the normal cone is given to graph of the normal cone constrained by generalized polyhedra without any constraint qualifications. Next, under the auxiliary polyhedral set and the provided constraint qualification, the formula of the above-obtained normal cone is simplified. Finally, the coderivative of the solution mapping to the parametric variational inequality is given.

Key words: parametric variational inequalities; generalized polyhedra; coderivative