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Global exponential stability of cycle associative neural network with constant delays

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Abstract: The global exponential stability of cycle associative neural network with constant delays is discussed. During the discussion, by constructing homeomorphism mapping, it is demonstrated that there exists an equilibrium point which is unique for this system, then the global exponential stability of the unique equilibrium point is testified by constructing proper Lyapunov function. Similar to previous work about neural network stability, under the assumption that the activation function about neuron satisfies Lipschitz condition and the matrix constructed by correlation coefficient satisfies given condition, the dynamics of global exponential stability for n-layer neural network with constant delays are obtained. The results contain that when the passive rate of neuron is sufficiently large, the neural network is global exponential stable.

Key words: exponential stability; equilibrium point; neural network; Lyapunov function

0 Introduction

The dynamical behaviors of delayed neural networks have attracted increasing interest for their intense application. Especially, there are many works about stability of neural network[1-8]. In Lits. [2-3], the authors discussed the static network with S-type distributed delays. In Lit. [4], the author discussed the global exponential stability of a class of neural networks with delays by nature M-matrix. In Lits. [3-5], the authors discussed the global exponential stability of the one-layer neural network. At the same time, the stability of bidirectional associative memory neural networks of the two-layers with delays has also been studied by many researchers[6-8]. In Lits. [5-6] the authors discussed the existence of equilibrium point and the global exponential stability by homeomorphism and constructing proper Lyapunov function. Inspired by above work, we should discuss the exponential stability of n-layers neural networks with constant delays, which should be taken as general form for work[6].

1 Model and preliminaries

In this paper, we should discuss the cycle associative neural network of the n-layers with constant delays:

$$\dot{u}_{i_1,1}(t)=-a_{i_1,1}u_{i_1,1}(t)+\sum_{i_2=1}^{l_2}w_{1,i_1,i_2}s_{i_2}^{(1)}(u_{i_2,2}(t-\tau_1))+J_{i_1}^{(1)};\quad i_1=1,2,\cdots,l_1$$

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$$\dot{u}_{i_2,2}(t)=-a_{i_2,2}u_{i_2,2}(t)+\sum_{i_3=1}^{l_3}w_{2,i_2,i_3}s_{i_3}^{(2)}(u_{i_3,3}(t-\tau_2))+J_{i_2}^{(2)};\quad i_2=1,2,\cdots,l_2$$

$$\cdots$$

$$\dot{u}_{i_n,n}(t)=-a_{i_n,n}u_{i_n,n}(t)+\sum_{i_1=1}^{l_1}w_{n,i_n,i_1}s_{i_1}^{(n)}(u_{i_1,1}(t-\tau_n))+J_{i_n}^{(n)};\quad i_n=1,2,\cdots,l_n \tag{1}$$

where $a_{i_1,1}$, $a_{i_2,2}$, $\cdots$, $a_{i_n,n}$; w_{1,i_1,i_2}, w_{2,i_2,i_3}, $\cdots$, w_{n,i_n,i_1}; $J_{i_1}^{(1)}$, $J_{i_2}^{(2)}$, $\cdots$, $J_{i_n}^{(n)}$; $s_{i_2}^{(1)}$, $s_{i_3}^{(2)}$, $\cdots$, $s_{i_1}^{(n)}$ have the same explanation to a_i, b_j; w_{ij}, v_{ji}; $J_i^{(1)}$, $J_j^{(2)}$; $s_i^{(1)}$, $s_j^{(2)}$ of system (2.1) in Lit. [6], similar to Lit. [6], the active function satisfies the assumption: There exist some positive constants $\alpha_{i_2}^{(1)}$, $i_2=1,2,\cdots,l_2$, $\alpha_{i_3}^{(2)}$, $i_3=1,2,\cdots,l_3$, $\cdots$, $\alpha_{i_1}^{(n)}$, $i_1=1,2,\cdots,l_1$, such that

$$0\leqslant\frac{s_{i_2}^{(1)}(x)-s_{i_2}^{(1)}(y)}{x-y}\leqslant\alpha_{i_2}^{(1)};$$
$$i_2=1,2,\cdots,l_2,\ \forall x,y\in\mathbf{R},\ x\neq y$$
$$0\leqslant\frac{s_{i_3}^{(2)}(x)-s_{i_3}^{(2)}(y)}{x-y}\leqslant\alpha_{i_3}^{(2)};$$
$$i_3=1,2,\cdots,l_3,\ \forall x,y\in\mathbf{R},\ x\neq y \tag{2}$$
$$\cdots$$
$$0\leqslant\frac{s_{i_1}^{(n)}(x)-s_{i_1}^{(n)}(y)}{x-y}\leqslant\alpha_{i_1}^{(n)};$$
$$i_1=1,2,\cdots,l_1,\ \forall x,y\in\mathbf{R},\ x\neq y$$

Let $\tau=\max(\tau_1,\tau_2,\cdots,\tau_n)$, initial conditions for network (1) are of the form

$$\boldsymbol{\phi}=(\phi_1\quad\cdots\quad\phi_{l_1}\ \phi_{l_1+1}\quad\cdots\quad\phi_{l_1+l_2}\quad\cdots\quad\phi_{l_1+l_2+\cdots+l_{n-1}+1}\quad\cdots\quad\phi_{l_1+l_2+\cdots+l_n})\in\boldsymbol{C}=\boldsymbol{C}([-\tau,0],\boldsymbol{R}^{l_1+l_2+\cdots+l_n})$$

Where $\boldsymbol{C}$ is the continuous function space with the norm $\|\boldsymbol{\phi}\|_2=\sup\limits_{-\tau\leqslant t\leqslant 0}\boldsymbol{\phi}^{\mathrm{T}}(t)\boldsymbol{\phi}(t)$. Network (1) admits a unique solution under $\boldsymbol{\phi}$, we denote

$$(\boldsymbol{u}_1(t,\boldsymbol{\phi}),\boldsymbol{u}_2(t,\boldsymbol{\phi}),\cdots,\boldsymbol{u}_n(t,\boldsymbol{\phi}))=(u_{1,1}(t,\boldsymbol{\phi})\quad\cdots\quad u_{l_1,1}(t,\boldsymbol{\phi})\quad u_{1,2}(t,\boldsymbol{\phi})\quad\cdots\quad u_{l_2,2}(t,\boldsymbol{\phi})\quad\cdots\quad u_{1,n}(t,\boldsymbol{\phi})\quad\cdots\quad u_{l_n,n}(t,\boldsymbol{\phi}))$$

For simplifying, the dependence on the initial condition $\boldsymbol{\phi}$ will not be indicated unless necessary. In the following, $\boldsymbol{A}$, $\boldsymbol{A}^{\mathrm{T}}$, $\boldsymbol{A}^{-1}$, $\boldsymbol{A}>\boldsymbol{0}$ $(<\boldsymbol{0})$, $\|\boldsymbol{A}\|_2$, v, $f(v)$ have the same definition as in Lit. [6].

Denote $\boldsymbol{x}=(\boldsymbol{u}_1\quad\boldsymbol{u}_2\quad\cdots\quad\boldsymbol{u}_n)=(u_{1,1}\quad\cdots\quad u_{l_1,1}\quad u_{1,2}\quad\cdots\quad u_{l_2,2}\quad\cdots\quad u_{1,n}\quad\cdots\quad u_{l_n,n})$. Hence, we write network (1) as

$$\dot{\boldsymbol{u}}_1^{\mathrm{T}}=-\boldsymbol{A}_1\boldsymbol{u}_1^{\mathrm{T}}+\boldsymbol{W}_1\boldsymbol{S}^{(1)}(\boldsymbol{u}_2(t-\tau_1))+\boldsymbol{J}^{(1)}$$
$$\dot{\boldsymbol{u}}_2^{\mathrm{T}}=-\boldsymbol{A}_2\boldsymbol{u}_2^{\mathrm{T}}+\boldsymbol{W}_2\boldsymbol{S}^{(2)}(\boldsymbol{u}_3(t-\tau_2))+\boldsymbol{J}^{(2)}$$
$$\cdots$$
$$\dot{\boldsymbol{u}}_n^{\mathrm{T}}=-\boldsymbol{A}_n\boldsymbol{u}_n^{\mathrm{T}}+\boldsymbol{W}_n\boldsymbol{S}^{(n)}(\boldsymbol{u}_1(t-\tau_n))+\boldsymbol{J}^{(n)}$$

where $\boldsymbol{A}_1=\mathbf{diag}\{a_{1,1},\cdots,a_{l_1,1}\}$, $\boldsymbol{A}_2=\mathbf{diag}\{a_{1,2},\cdots,a_{l_2,2}\}$, $\cdots$, $\boldsymbol{A}_n=\mathbf{diag}\{a_{1,n},\cdots,a_{l_n,n}\}$, and $\boldsymbol{W}_1=(w_{1,i_1,i_2})_{l_1\times l_2}$, $\boldsymbol{W}_2=(w_{2,i_2,i_3})_{l_2\times l_3}$, $\cdots$, $\boldsymbol{W}_n=(w_{n,i_n,i_1})_{l_n\times l_1}$.

The definition of global exponential stability for equilibrium point $\boldsymbol{x}^*=(\boldsymbol{u}_1^*\quad\boldsymbol{u}_2^*\quad\cdots\quad\boldsymbol{u}_n^*)$ of network (1) is same to definition 2.1 in Lit. [6], in order to assure the global exponential stability for equilibrium point $\boldsymbol{x}^*=(\boldsymbol{u}_1^*\quad\boldsymbol{u}_2^*\quad\cdots\quad\boldsymbol{u}_n^*)$ of network (1), we give the following conditions:

For network (1) there exist positive diagonal matrices $\boldsymbol{P}_1=\mathbf{diag}\{p_1^{(1)},\cdots,p_{l_1}^{(1)}\}$, $\boldsymbol{P}_2=\mathbf{diag}\{p_1^{(2)},\cdots,p_{l_2}^{(2)}\}$, $\cdots$, $\boldsymbol{P}_n=\mathbf{diag}\{p_1^{(n)},\cdots,p_{l_n}^{(n)}\}$, positive definite matrices $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, $\cdots$, $\boldsymbol{Q}_n$ and factorization of $\boldsymbol{W}_1=\boldsymbol{W}_{11}\boldsymbol{W}_{12}$, $\boldsymbol{W}_2=\boldsymbol{W}_{21}\boldsymbol{W}_{22}$, $\cdots$, $\boldsymbol{W}_n=\boldsymbol{W}_{n1}\boldsymbol{W}_{n2}$ such that the condition (T) holds

$$\text{(T)}\quad\begin{aligned}&\boldsymbol{\Omega}_1=2\boldsymbol{P}_2\boldsymbol{A}_2(\boldsymbol{\alpha}^{(1)})^{-1}-\boldsymbol{P}_2\boldsymbol{W}_{21}\boldsymbol{Q}_2^{-1}\boldsymbol{W}_{21}^{\mathrm{T}}\boldsymbol{P}_2-\boldsymbol{W}_{12}^{\mathrm{T}}\boldsymbol{Q}_1\boldsymbol{W}_{12}>\boldsymbol{0}\\&\boldsymbol{\Omega}_2=2\boldsymbol{P}_3\boldsymbol{A}_3(\boldsymbol{\alpha}^{(2)})^{-1}-\boldsymbol{P}_3\boldsymbol{W}_{31}\boldsymbol{Q}_3^{-1}\boldsymbol{W}_{31}^{\mathrm{T}}\boldsymbol{P}_3-\boldsymbol{W}_{22}^{\mathrm{T}}\boldsymbol{Q}_2\boldsymbol{W}_{22}>\boldsymbol{0}\\&\cdots\\&\boldsymbol{\Omega}_n=2\boldsymbol{P}_1\boldsymbol{A}_1(\boldsymbol{\alpha}^{(n)})^{-1}-\boldsymbol{P}_1\boldsymbol{W}_{11}\boldsymbol{Q}_1^{-1}\boldsymbol{W}_{11}^{\mathrm{T}}\boldsymbol{P}_1-\boldsymbol{W}_{n2}^{\mathrm{T}}\boldsymbol{Q}_n\boldsymbol{W}_{n2}>\boldsymbol{0}\end{aligned}$$

where $(\boldsymbol{\alpha}^{(1)})^{-1}=\mathbf{diag}\{(\alpha_1^{(1)})^{-1},\cdots,(\alpha_{l_2}^{(1)})^{-1}\}$, $(\boldsymbol{\alpha}^{(2)})^{-1}=\mathbf{diag}\{(\alpha_1^{(2)})^{-1},\cdots,(\alpha_{l_3}^{(2)})^{-1}\}$, $\cdots$, $(\boldsymbol{\alpha}^{(n)})^{-1}=\mathbf{diag}\{(\alpha_1^{(n)})^{-1},\cdots,(\alpha_{l_1}^{(n)})^{-1}\}$; $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, $\cdots$, $\boldsymbol{Q}_n$, $\boldsymbol{W}_{11}$, $\boldsymbol{W}_{12}$, $\boldsymbol{W}_{21}$, $\boldsymbol{W}_{22}$, $\cdots$, $\boldsymbol{W}_{n1}$, $\boldsymbol{W}_{n2}$ are constant matrices with appropriate dimension. Now we give the main results of this paper.

Theorem 1 For network (1), the assumption (2) and condition (T) hold. Then the neural network (1) has a unique equilibrium point.

Theorem 2 For network (1), the assumption (2) and condition (T) hold. Then the equilibrium point of neural network (1) is global exponential stable.

We should discuss the existence and uniqueness of the equilibrium point, the global exponential stability, and compare our result with previous results and give an example.

2 The existence and uniqueness of equilibrium point

For convenience we state the following lemma, which is special case of lemma (2.1) in Lit. [6].

Lemma 1 Given any real vectors $\boldsymbol{X},\boldsymbol{Y}$ of appropriate dimensions, then the following inequality holds

$$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}\leqslant\frac{\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}+\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Y}}{2}$$

Proof of Theorem 1 Let $\boldsymbol{x}^*=(\boldsymbol{u}_1^* \quad \boldsymbol{u}_2^* \quad \cdots \quad \boldsymbol{u}_n^*)$ be an equilibrium point of network (1). Then $\boldsymbol{x}^*$ satisfies

$$\begin{aligned}&-\boldsymbol{A}_1\boldsymbol{u}_1^{*\mathrm{T}}+\boldsymbol{W}_1\boldsymbol{S}^{(1)}(\boldsymbol{u}_2^{*\mathrm{T}})+\boldsymbol{J}^{(1)}=\boldsymbol{0}\\&-\boldsymbol{A}_2\boldsymbol{u}_2^{*\mathrm{T}}+\boldsymbol{W}_2\boldsymbol{S}^{(2)}(\boldsymbol{u}_3^{*\mathrm{T}})+\boldsymbol{J}^{(2)}=\boldsymbol{0}\\&\cdots\\&-\boldsymbol{A}_n\boldsymbol{u}_n^{*\mathrm{T}}+\boldsymbol{W}_n\boldsymbol{S}^{(n)}(\boldsymbol{u}_1^{*\mathrm{T}})+\boldsymbol{J}^{(n)}=\boldsymbol{0}\end{aligned}$$

Let

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{x})=&(\boldsymbol{H}_1(\boldsymbol{x}) \quad \boldsymbol{H}_2(\boldsymbol{x}) \quad \cdots \quad \boldsymbol{H}_n(\boldsymbol{x}))^{\mathrm{T}}=\\&(-\boldsymbol{A}_1\boldsymbol{u}_1^{\mathrm{T}}+\boldsymbol{W}_1\boldsymbol{S}^{(1)}(\boldsymbol{x})+\boldsymbol{J}^{(1)}\\&-\boldsymbol{A}_2\boldsymbol{u}_2^{\mathrm{T}}+\boldsymbol{W}_2\boldsymbol{S}^{(2)}(\boldsymbol{x})+\boldsymbol{J}^{(2)} \quad \cdots\\&-\boldsymbol{A}_n\boldsymbol{u}_n^{\mathrm{T}}+\boldsymbol{W}_n\boldsymbol{S}^{(n)}(\boldsymbol{x})+\boldsymbol{J}^{(n)})^{\mathrm{T}}=\boldsymbol{0} \qquad (3)\end{aligned}$$

where $\boldsymbol{S}^{(1)}(\boldsymbol{x})=(s_1^{(1)}(u_{1,2}) \quad \cdots \quad s_{l_2}^{(1)}(u_{l_2,2}))^{\mathrm{T}}$, $\boldsymbol{S}^{(2)}(\boldsymbol{x})=(s_1^{(2)}(u_{1,3}) \quad \cdots \quad s_{l_3}^{(2)}(u_{l_3,3}))^{\mathrm{T}}$, $\cdots$, $\boldsymbol{S}^{(n)}(\boldsymbol{x})=(s_1^{(n)}(u_{1,1}) \quad \cdots \quad s_{l_1}^{(n)}(u_{l_1,1}))^{\mathrm{T}}$.

From Lit. [9], if $\boldsymbol{H}(\boldsymbol{x})$ satisfies $\boldsymbol{H}(\boldsymbol{x})\neq\boldsymbol{H}(\boldsymbol{y})$, $\forall\, \boldsymbol{x}\neq\boldsymbol{y}$, $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{R}^{l_1+l_2+\cdots+l_n}$, and $\|\boldsymbol{H}(\boldsymbol{x})\|\rightarrow\infty$, as $\|\boldsymbol{x}\|\rightarrow\infty$, then $\boldsymbol{H}(\boldsymbol{x})$ is a homeomorphism of $\boldsymbol{R}^{l_1+l_2+\cdots+l_n}$. Because the solution of Eq. (3) is equilibrium point of network (1), so we demonstrate that network (1) has a unique equilibrium point by demonstrating $\boldsymbol{H}(\boldsymbol{x})$ is a homeomorphism of $\boldsymbol{R}^{l_1+l_2+\cdots+l_n}$.

Let $\boldsymbol{S}(\boldsymbol{x})=(\boldsymbol{S}^{(n)}(\boldsymbol{x}) \quad \boldsymbol{S}^{(1)}(\boldsymbol{x}) \quad \cdots \quad \boldsymbol{S}^{(n-1)}(\boldsymbol{x}))^{\mathrm{T}}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ be two vectors such that $\boldsymbol{x}\neq\boldsymbol{y}$. Under the assumption (2) on the activation functions $\boldsymbol{x}\neq\boldsymbol{y}$ imply two cases: (i) $\boldsymbol{x}\neq\boldsymbol{y}$ and $\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{y})\neq\boldsymbol{0}$; (ii) $\boldsymbol{x}\neq\boldsymbol{y}$ and $\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{y})=\boldsymbol{0}$, now we write

$$\begin{aligned}\boldsymbol{H}_1(\boldsymbol{x})-\boldsymbol{H}_1(\boldsymbol{y})=&-\boldsymbol{A}_1\boldsymbol{u}_{1x}+\boldsymbol{A}_1\boldsymbol{u}_{1y}+\boldsymbol{W}_1(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\\&\boldsymbol{S}^{(1)}(\boldsymbol{y}))\\\boldsymbol{H}_2(\boldsymbol{x})-\boldsymbol{H}_2(\boldsymbol{y})=&-\boldsymbol{A}_2\boldsymbol{u}_{2x}+\boldsymbol{A}_2\boldsymbol{u}_{2y}+\boldsymbol{W}_2(\boldsymbol{S}^{(2)}(\boldsymbol{x})-\\&\boldsymbol{S}^{(2)}(\boldsymbol{y}))\\&\cdots\\\boldsymbol{H}_n(\boldsymbol{x})-\boldsymbol{H}_n(\boldsymbol{y})=&-\boldsymbol{A}_n\boldsymbol{u}_{nx}+\boldsymbol{A}_n\boldsymbol{u}_{ny}+\boldsymbol{W}_n(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\\&\boldsymbol{S}^{(n)}(\boldsymbol{y})) \qquad (4)\end{aligned}$$

where $\boldsymbol{u}_{1x}=(u_{1,1x} \quad \cdots \quad u_{l_1,1x})^{\mathrm{T}}$, $\boldsymbol{u}_{1y}=(u_{1,1y} \quad \cdots \quad u_{l_1,1y})^{\mathrm{T}}$, $\boldsymbol{u}_{2x}=(u_{1,2x} \quad \cdots \quad u_{l_2,2x})^{\mathrm{T}}$, $\boldsymbol{u}_{2y}=(u_{1,2y} \quad \cdots \quad u_{l_2,2y})^{\mathrm{T}}$, $\cdots$, $\boldsymbol{u}_{nx}=(u_{1,nx} \quad \cdots \quad u_{l_n,nx})^{\mathrm{T}}$, $\boldsymbol{u}_{ny}=(u_{1,ny} \quad \cdots \quad u_{l_n,ny})^{\mathrm{T}}$.

First, we consider the case (i). In this case, there exists $k\in(1,2,\cdots,n)$ such that $\boldsymbol{S}^{(k)}(\boldsymbol{x})\neq\boldsymbol{S}^{(k)}(\boldsymbol{y})$. Multiplying both sides of the first equation in Eq. (4) by $2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1$, results in

$$\begin{aligned}&2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1(\boldsymbol{H}_1(\boldsymbol{x})-\boldsymbol{H}_1(\boldsymbol{y}))=\\&-2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1(\boldsymbol{A}_1\boldsymbol{u}_{1x}-\boldsymbol{A}_1\boldsymbol{u}_{1y})+\\&2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{W}_1(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))\end{aligned}$$

since $(u_{i_1,1x}-u_{i_1,1y})(\mathrm{S}_i^{(n)}(u_{i_1,1x})-\mathrm{S}_i^{(n)}(u_{i_1,1y}))\geqslant\left(\frac{1}{\alpha_{i_1}^{(n)}}\right)(\mathrm{S}_{i_1}^{(n)}(u_{i_1,1x})-\mathrm{S}_{i_1}^{(n)}(u_{i_1,1y}))^2$

we have

$$\begin{aligned}&(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1(\boldsymbol{A}_1\boldsymbol{u}_{1x}-\boldsymbol{A}_1\boldsymbol{u}_{1y})\geqslant\\&(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{A}_1(\boldsymbol{\alpha}^{(n)})^{-1}(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\\&\boldsymbol{S}^{(n)}(\boldsymbol{y}))\end{aligned}$$

Let the Cholesky factorization of $\boldsymbol{Q}_1$ be $\boldsymbol{Q}_1=\boldsymbol{K}_1^{\mathrm{T}}\boldsymbol{K}_1$. Rewriting $\boldsymbol{W}_1=(\boldsymbol{W}_{11}\boldsymbol{K}_1^{-1})(\boldsymbol{K}_1\boldsymbol{W}_{12})$, we have

$$\begin{aligned}&2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1(\boldsymbol{H}_1(\boldsymbol{x})-\boldsymbol{H}_1(\boldsymbol{y}))\leqslant\\&-2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{A}_1(\boldsymbol{\alpha}^{(n)})^{-1}(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\\&\boldsymbol{S}^{(n)}(\boldsymbol{y}))+2[(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{W}_{11}\boldsymbol{K}_1^{-1}]\times\\&[\boldsymbol{K}_1\boldsymbol{W}_{12}(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))]\end{aligned}$$

It follows from Lemma 1

$$\begin{aligned}&2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1(\boldsymbol{H}_1(\boldsymbol{x})-\boldsymbol{H}_1(\boldsymbol{y}))\leqslant\\&-2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{A}_1(\boldsymbol{\alpha}^{(n)})^{-1}(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\\&\boldsymbol{S}^{(n)}(\boldsymbol{y}))+(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{W}_{11}\boldsymbol{Q}_1^{-1}\times\\&\boldsymbol{W}_{11}^{\mathrm{T}}\boldsymbol{P}_1(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))+(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\\&\boldsymbol{S}^{(1)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{W}_{12}^{\mathrm{T}}\boldsymbol{Q}_1\boldsymbol{W}_{12}(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y})) \qquad (5)\end{aligned}$$

Similarly

$$\begin{aligned}&2(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_2(\boldsymbol{H}_2(\boldsymbol{x})-\boldsymbol{H}_2(\boldsymbol{y}))\leqslant\\&-2(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{A}_2(\boldsymbol{\alpha}^{(1)})^{-1}(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\\&\boldsymbol{S}^{(1)}(\boldsymbol{y}))+(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{W}_{21}\boldsymbol{Q}_2^{-1}\times\end{aligned}$$

$$\boldsymbol{W}_{21}^{\mathrm{T}}\boldsymbol{P}_2(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))+(\boldsymbol{S}^{(2)}(\boldsymbol{x})-\boldsymbol{S}^{(2)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{W}_{22}^{\mathrm{T}}\boldsymbol{Q}_2\boldsymbol{W}_{22}(\boldsymbol{S}^{(2)}(\boldsymbol{x})-\boldsymbol{S}^{(2)}(\boldsymbol{y})) \tag{6}$$

$$\cdots$$

$$2(\boldsymbol{S}^{(n-1)}(\boldsymbol{x})-\boldsymbol{S}^{(n-1)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_n(\boldsymbol{H}_n(\boldsymbol{x})-\boldsymbol{H}_n(\boldsymbol{y}))\leqslant -2(\boldsymbol{S}^{(n-1)}(\boldsymbol{x})-\boldsymbol{S}^{(n-1)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_n\boldsymbol{A}_n(\boldsymbol{\alpha}^{(n-1)})^{-1}\times (\boldsymbol{S}^{(n-1)}(\boldsymbol{x})-\boldsymbol{S}^{(n-1)}(\boldsymbol{y}))+(\boldsymbol{S}^{(n-1)}(\boldsymbol{x})-\boldsymbol{S}^{(n-1)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{P}_3\boldsymbol{W}_{n1}\boldsymbol{Q}_n^{-1}\boldsymbol{W}_{n1}^{\mathrm{T}}\boldsymbol{P}_n(\boldsymbol{S}^{(n-1)}(\boldsymbol{x})-\boldsymbol{S}^{(n-1)}(\boldsymbol{y}))+(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{W}_{n2}^{\mathrm{T}}\boldsymbol{Q}_n\boldsymbol{W}_{n2}\times (\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y})) \tag{7}$$

which imply that

$$\boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y})=(2(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\quad 2(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))^{\mathrm{T}}\quad\cdots\quad 2(\boldsymbol{S}^{(n-1)}(\boldsymbol{x})-\boldsymbol{S}^{(n-1)}(\boldsymbol{y}))^{\mathrm{T}})\mathbf{diag}\{\boldsymbol{P}_1,\boldsymbol{P}_2,\cdots,\boldsymbol{P}_n\}\times (\boldsymbol{H}(\boldsymbol{x})-\boldsymbol{H}(\boldsymbol{y}))$$

Noting that $\boldsymbol{Q}_2=\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{K}_2,\cdots,\boldsymbol{Q}_n=\boldsymbol{K}_n^{\mathrm{T}}\boldsymbol{K}_n$, from condition (T) we obtain that

$$\boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{y})\leqslant-(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{\Omega}_1(\boldsymbol{S}^{(1)}(\boldsymbol{x})-\boldsymbol{S}^{(1)}(\boldsymbol{y}))-(\boldsymbol{S}^{(2)}(\boldsymbol{x})-\boldsymbol{S}^{(2)}(\boldsymbol{y}))^{\mathrm{T}}\times \boldsymbol{\Omega}_2(\boldsymbol{S}^{(2)}(\boldsymbol{x})-\boldsymbol{S}^{(2)}(\boldsymbol{y}))-\cdots-(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{\Omega}_n(\boldsymbol{S}^{(n)}(\boldsymbol{x})-\boldsymbol{S}^{(n)}(\boldsymbol{y}))<\boldsymbol{0} \tag{8}$$

That is $\boldsymbol{H}(\boldsymbol{x})\neq\boldsymbol{H}(\boldsymbol{y})$. Since $\mathbf{diag}\{\boldsymbol{P}_1,\boldsymbol{P}_2,\cdots,\boldsymbol{P}_n\}$ is a positive diagonal matrix, we prove that $\boldsymbol{H}(\boldsymbol{x})-\boldsymbol{H}(\boldsymbol{y})\neq\boldsymbol{0}$ when $\boldsymbol{x}\neq\boldsymbol{y}$ and $\boldsymbol{S}(\boldsymbol{x})\neq\boldsymbol{S}(\boldsymbol{y})$.

Now we consider the case (ii). In view of $\boldsymbol{x}\neq\boldsymbol{y}$ and $\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{y})=\boldsymbol{0}$, we have

$$\boldsymbol{H}(\boldsymbol{x})-\boldsymbol{H}(\boldsymbol{y})=-\begin{pmatrix}\boldsymbol{A}_1&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{A}_2&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\ddots&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{A}_n\end{pmatrix}(\boldsymbol{x}-\boldsymbol{y})\neq\boldsymbol{0}$$

which implies that $\boldsymbol{H}(\boldsymbol{x})\neq\boldsymbol{H}(\boldsymbol{y})$ for $\boldsymbol{x}\neq\boldsymbol{y}$.

In the following, we claim that the condition of Theorem 1 also implies that $\|\boldsymbol{H}(\boldsymbol{x})\|\rightarrow\infty$, as $\|\boldsymbol{x}\|\rightarrow\infty$. Setting $\eta=0$, inequality (8) yields

$$\boldsymbol{\Psi}(\boldsymbol{x},\boldsymbol{0})\leqslant-\lambda_{\min}[(\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{0}))^{\mathrm{T}}(\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{0}))]$$

where $\lambda_{\min}$ denotes the minimum eigenvalue of the positive definite matrices $\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2,\cdots,\boldsymbol{\Omega}_n$. Similar to Lemma 2.2 in Lit. [6], we obtain

$$\lambda_{\min}\|\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{0})\|_2^2\leqslant 2p\|\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{0})\|_\infty\times\|\boldsymbol{H}(\boldsymbol{x})-\boldsymbol{H}(\boldsymbol{0})\|_1$$

where $p=\max(p_1^{(1)},\cdots,p_{l_1}^{(1)},p_1^{(2)},\cdots,p_{l_2}^{(2)},\cdots,p_1^{(n)},\cdots,p_{l_n}^{(n)})$, using the fact $\|\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{0})\|_\infty\leqslant\|\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{0})\|_2$, $\|\boldsymbol{S}(\boldsymbol{x})\|_\infty-\|\boldsymbol{S}(\boldsymbol{0})\|_\infty\leqslant\|\boldsymbol{S}(\boldsymbol{x})-\boldsymbol{S}(\boldsymbol{0})\|_\infty$ and $\|\boldsymbol{H}(\boldsymbol{x})-\boldsymbol{H}(\boldsymbol{0})\|_1\leqslant\|\boldsymbol{H}(\boldsymbol{x})\|_1+\|\boldsymbol{H}(\boldsymbol{0})\|_1$, we have

$$\lambda_{\min}\|\boldsymbol{S}(\boldsymbol{x})\|_\infty-\lambda_{\min}\|\boldsymbol{S}(\boldsymbol{0})\|_\infty\leqslant 2p\|\boldsymbol{H}(\boldsymbol{x})\|_1+2p\|\boldsymbol{H}(\boldsymbol{0})\|_1$$

Hence

$$\|\boldsymbol{H}(\boldsymbol{x})\|_1\geqslant(\lambda_{\min}\|\boldsymbol{S}(\boldsymbol{x})\|_\infty-\lambda_{\min}\|\boldsymbol{S}(\boldsymbol{0})\|_\infty-2p\|\boldsymbol{H}(\boldsymbol{0})\|_1)/2p$$

which implies $\|\boldsymbol{H}(\boldsymbol{x})\|\rightarrow\infty$, as $\|\boldsymbol{S}(\boldsymbol{x})\|\rightarrow\infty$. So we conclude $\|\boldsymbol{H}(\boldsymbol{x})\|\rightarrow\infty$, as $\|\boldsymbol{x}\|\rightarrow\infty$. Hence we prove that $\boldsymbol{H}(\boldsymbol{x})$ is homeomorphism of $\boldsymbol{R}^{l_1+l_2+\cdots+l_n}$, which ensures that the neural network (1) has an equilibrium point and the equilibrium point is unique.

3 The global exponential stability of the equilibrium point

First, we shift the equilibrium point of network (1) to the origin by the transformation $\boldsymbol{v}_1(t)=\boldsymbol{u}_1(t)-\boldsymbol{u}_1^*$, $\boldsymbol{v}_2(t)=\boldsymbol{u}_2(t)-\boldsymbol{u}_2^*$, $\cdots$, $\boldsymbol{v}_n(t)=\boldsymbol{u}_n(t)-\boldsymbol{u}_n^*$, network (1) can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{v}}_1(\mathrm{t})&=-\boldsymbol{A}_1\boldsymbol{v}_1(t)+\boldsymbol{W}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))\\ \dot{\boldsymbol{v}}_2(t)&=-\boldsymbol{A}_2\boldsymbol{v}_2(t)+\boldsymbol{W}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))\\ &\cdots\\ \dot{\boldsymbol{v}}_n(t)&=-\boldsymbol{A}_n\boldsymbol{v}_n(t)+\boldsymbol{W}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))\end{aligned} \tag{9}$$

where $\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t))=(f_1^{(1)}(v_{1,2}(t))\ \cdots\ f_{l_2}^{(1)}(v_{l_2,2}(t)))$, $\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t))=(f_1^{(2)}(v_{1,3}(t))\ \cdots\ f_{l_3}^{(2)}(v_{l_3,3}(t)))$, $\cdots$, $\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t))=(f_1^{(n)}(v_{1,1}(t))\ \cdots\ f_{l_n}^{(n)}(v_{l_1,1}(t)))$. From Eq. (9), we get

$$f_{i_2}^{(1)}(v_{i_2,2}(t))=s_{i_2}^{(1)}(v_{i_2,2}(t)+u_{i_2,2}^*)-s_{i_2}^{(1)}(u_{i_2,2}^*);\quad i_2=1,2,\cdots,l_2$$

$$f_{i_3}^{(2)}(v_{i_3,3}(t))=s_{i_3}^{(2)}(v_{i_3,3}(t)+u_{i_3,3}^*)-s_{i_3}^{(2)}(u_{i_3,3}^*);\quad i_3=1,2,\cdots,l_3$$

$$\cdots$$

$$f_{i_1}^{(n)}(v_{i_1,1}(t))=s_{i_1}^{(n)}(v_{i_1,1}(t)+u_{i_1,1}^*)-s_{i_1}^{(n)}(u_{i_1,1}^*);\quad i_1=1,2,\cdots,l_1$$

The Lipschitz condition implies that

$$0\leqslant\frac{f_{i_2}^{(1)}(v_{i_2,2}(t))}{v_{i_2,2}(t)}\leqslant\alpha_{i_2}^{(1)};\quad f_{i_2}^{(1)}(0)=0,\ i_2=1,2,\cdots,l_2$$

$$0\leqslant\frac{f_{i_3}^{(2)}(v_{i_3,3}(t))}{v_{i_3,3}(t)}\leqslant\alpha_{i_3}^{(2)};\quad f_{i_3}^{(2)}(0)=0,\ i_3=1,2,\cdots,l_3$$

$$\cdots$$

$$0 \leqslant \frac{f_{i_1}^{(n)}(v_{i_1,1}(t))}{v_{i_1,1}(t)} \leqslant \alpha_{i_1}^{(n)};$$
$$f_{i_1}^{(n)}(0)=0,\ i_1=1,2,\cdots,l_1$$

Obviously $\boldsymbol{x}^*=(\boldsymbol{u}_1^* \quad \boldsymbol{u}_2^* \quad \cdots \quad \boldsymbol{u}_n^*)$ of network (1) is the global exponential stable if and only if the origin of Eq. (9) is global exponential stable. Now we consider the origin of Eq. (9).

Proof of Theorem 2 We employ the following Lyapunov function

$$\begin{aligned}&V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)=\\&\varepsilon_1 V_1(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t))+\\&V_2(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)\end{aligned} \tag{10}$$

where

$$\begin{aligned}&V_1(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t))=\\&\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{v}_1(t)+\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{v}_2(t)+\cdots+\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{v}_n(t)\\&V_2(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)=\\&2\sum_{i_2=1}^{l_2}p_{i_2}^{(2)}\int_0^{v_{i_2,2}(t)}f_{i_2}^{(1)}(s)\mathrm{d}s+\\&\int_{t-\tau_1}^{t}\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(\tau))\boldsymbol{R}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(\tau))\mathrm{d}\tau+\\&2\sum_{i_3=1}^{l_3}p_{i_3}^{(3)}\int_0^{v_{i_3,3}(t)}f_{i_3}^{(2)}(s)\mathrm{d}s+\\&\int_{t-\tau_2}^{t}\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(\tau))\boldsymbol{R}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(\tau))\mathrm{d}\tau+\cdots+\\&2\sum_{i_1=1}^{l_1}p_{i_1}^{(1)}\int_0^{v_{i_1,1}(t)}f_{i_1}^{(n)}(s)\mathrm{d}s+\\&\int_{t-\tau_n}^{t}\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(\tau))\boldsymbol{R}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(\tau))\mathrm{d}\tau\end{aligned}$$

First we compute the derivative of V along trajectories of Eq. (9), then determine positive constant ε_1 and positive definite matrices $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\cdots$, $\boldsymbol{R}_n$.

$$\begin{aligned}&\dot V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)=\\&\varepsilon_1\dot V_1(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\ \cdots,\ \boldsymbol{v}_n(t))+\\&\dot V_2(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\ \cdots,\ \boldsymbol{v}_n(t),t)\end{aligned}$$

where

$$\begin{aligned}&\dot V_1(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t))=\\&2\boldsymbol{v}_1^{\mathrm{T}}(t)[-\boldsymbol{A}_1\boldsymbol{v}_1(t)+\boldsymbol{W}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))]+\\&2\boldsymbol{v}_2^{\mathrm{T}}(t)[-\boldsymbol{A}_2\boldsymbol{v}_2(t)+\boldsymbol{W}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))]+\cdots+\\&2\boldsymbol{v}_n^{\mathrm{T}}(t)[-\boldsymbol{A}_n\boldsymbol{v}_n(t)+\boldsymbol{W}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))]\end{aligned}$$

and

$$\begin{aligned}&\dot V_2(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)=\\&2\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))\boldsymbol{P}_2[-\boldsymbol{A}_2\boldsymbol{v}_2(t)+\boldsymbol{W}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\\&\tau_2))]+2\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))\boldsymbol{P}_3[-\boldsymbol{A}_3\boldsymbol{v}_3(t)+\\&\boldsymbol{W}_3\boldsymbol{f}^{(3)}(\boldsymbol{v}_4(t-\tau_3))]+\cdots+\\&2\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))\boldsymbol{P}_1[-\boldsymbol{A}_1\boldsymbol{v}_1(t)+\boldsymbol{W}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))]+\\&\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))\boldsymbol{R}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t))-\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t-\tau_1))\times\\&\boldsymbol{R}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))+\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))\boldsymbol{R}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t))-\\&\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t-\tau_2))\boldsymbol{R}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))+\cdots+\\&\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))\boldsymbol{R}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t))-\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t-\tau_n))\times\\&\boldsymbol{R}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))\end{aligned}$$

Rewriting $\dot V_1$ as

$$\begin{aligned}&\dot V_1(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t))\leqslant\\&-2\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{A}_1\boldsymbol{v}_1(t)+2\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{A}_1^{1/2}\boldsymbol{A}_1^{-1/2}\boldsymbol{W}_1\times\\&\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))-2\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{A}_2\boldsymbol{v}_2(t)+\\&2\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{A}_2^{1/2}\boldsymbol{A}_2^{-1/2}\boldsymbol{W}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))+\cdots-\\&2\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{A}_n\boldsymbol{v}_n(t)+2\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{A}_n^{1/2}\boldsymbol{A}_n^{-1/2}\boldsymbol{W}_n\times\\&\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))\end{aligned}$$

It follows from Lemma 1 that

$$\begin{aligned}\dot V_1\leqslant&-\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{A}_1\boldsymbol{v}_1(t)-\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{A}_2\boldsymbol{v}_2(t)-\cdots-\\&\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{A}_n\boldsymbol{v}_n(t)+\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t-\tau_1))\times\\&\boldsymbol{W}_1^{\mathrm{T}}\boldsymbol{A}_1^{-1}\boldsymbol{W}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))+\\&\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t-\tau_2))\boldsymbol{W}_2^{\mathrm{T}}\boldsymbol{A}_2^{-1}\boldsymbol{W}_2\times\\&\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))+\cdots+\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t-\tau_n))\times\\&\boldsymbol{W}_n^{\mathrm{T}}\boldsymbol{A}_n^{-1}\boldsymbol{W}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))\leqslant\\&-\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{A}_1\boldsymbol{v}_1(t)-\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{A}_2\boldsymbol{v}_2(t)-\cdots-\\&\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{A}_n\boldsymbol{v}_n(t)+M[\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t-\tau_1))\times\\&\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))+\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t-\tau_2))\times\\&\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))+\cdots+\\&\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t-\tau_n))\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))]\end{aligned}$$

where $M=\max[\|\boldsymbol{W}_1^{\mathrm{T}}\boldsymbol{A}_1^{-1}\boldsymbol{W}_1\|_2,\ \|\boldsymbol{W}_2^{\mathrm{T}}\boldsymbol{A}_2^{-1}\boldsymbol{W}_2\|_2,\ \cdots,\ \|\boldsymbol{W}_n^{\mathrm{T}}\boldsymbol{A}_n^{-1}\boldsymbol{W}_n\|_2]\geqslant 0$. Since

$$f_{i_2}^{(1)}(v_{i_2,2}(t))v_{i_2,2}(t)\geqslant(\alpha_{i_2}^{(1)})^{-1}(f_{i_2}^{(1)}(v_{i_2,2}(t)))^2;$$
$$i_2=1,2,\cdots,l_2$$
$$f_{i_3}^{(2)}(v_{i_3,3}(t))v_{i_3,3}(t)\geqslant(\alpha_{i_3}^{(2)})^{-1}(f_{i_3}^{(2)}(v_{i_3,3}(t)))^2;$$
$$i_3=1,2,\cdots,l_3$$
$$\cdots$$
$$f_{i_1}^{(n)}(v_{i_1,1}(t))v_{i_1,1}(t)\geqslant(\alpha_{i_1}^{(n)})^{-1}(f_{i_1}^{(n)}(v_{i_1,1}(t)))^2;$$
$$i_1=1,2,\cdots,l_1$$

we get

$$\begin{aligned}&-\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))\boldsymbol{P}_2\boldsymbol{A}_2\boldsymbol{v}_2(t)\leqslant\\&-\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))\boldsymbol{P}_2\boldsymbol{A}_2(\boldsymbol{\alpha}^{(1)})^{-1}\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t))\\&-\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))\boldsymbol{P}_3\boldsymbol{A}_3\boldsymbol{v}_3(t)\leqslant\\&-\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))\boldsymbol{P}_3\boldsymbol{A}_3(\boldsymbol{\alpha}^{(2)})^{-1}\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t))\\&\cdots\\&-\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))\boldsymbol{P}_1\boldsymbol{A}_1\boldsymbol{v}_1(t)\leqslant\\&-\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))\boldsymbol{P}_1\boldsymbol{A}_1(\boldsymbol{\alpha}^{(n)})^{-1}\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t))\end{aligned}$$

Let the Cholesky factorization of $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, $\cdots$, $\boldsymbol{Q}_n$ be $\boldsymbol{Q}_1=\boldsymbol{K}_1^{\mathrm{T}}\boldsymbol{K}_1$, $\boldsymbol{Q}_2=\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{K}_2$, $\cdots$, $\boldsymbol{Q}_n=\boldsymbol{K}_n^{\mathrm{T}}\boldsymbol{K}_n$. Rewriting $\boldsymbol{W}_1=(\boldsymbol{W}_{11}\boldsymbol{K}_1^{-1})(\boldsymbol{K}_1\boldsymbol{W}_{12})$, $\boldsymbol{W}_2=(\boldsymbol{W}_{21}\boldsymbol{K}_2^{-1})(\boldsymbol{K}_2\boldsymbol{W}_{22})$, $\cdots$, $\boldsymbol{W}_n=(\boldsymbol{W}_{n1}\boldsymbol{K}_n^{-1})\times(\boldsymbol{K}_n\boldsymbol{W}_{n2})$, we have

$$\begin{aligned}\dot{V}_2\leqslant&-\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))2\boldsymbol{P}_2\boldsymbol{A}_2(\boldsymbol{\alpha}^{(1)})^{-1}\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t))-\\&\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))2\boldsymbol{P}_3\boldsymbol{A}_3(\boldsymbol{\alpha}^{(2)})^{-1}\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t))-\cdots-\\&\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))2\boldsymbol{P}_1\boldsymbol{A}_1(\boldsymbol{\alpha}^{(n)})^{-1}\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t))+\\&2(\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))\boldsymbol{P}_2(\boldsymbol{W}_{21}\boldsymbol{K}_2^{-1})(\boldsymbol{K}_2\boldsymbol{W}_{22})\times\\&\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2)))+2(\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))\times\\&\boldsymbol{P}_3(\boldsymbol{W}_{31}\boldsymbol{K}_3^{-1})(\boldsymbol{K}_3\boldsymbol{W}_{32})\boldsymbol{f}^{(3)}(\boldsymbol{v}_4(t-\tau_3)))+\cdots+\\&2(\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))\boldsymbol{P}_1(\boldsymbol{W}_{11}\boldsymbol{K}_1^{-1})(\boldsymbol{K}_1\boldsymbol{W}_{12})\times\\&\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1)))+\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))\times\\&\boldsymbol{R}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t))-\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t-\tau_1))\times\\&\boldsymbol{R}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))+\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))\times\\&\boldsymbol{R}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t))-\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t-\tau_2))\times\\&\boldsymbol{R}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))+\cdots+\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))\times\\&\boldsymbol{R}_n\boldsymbol{f}^{(1)}(\boldsymbol{v}_1(t))-\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t-\tau_n))\times\\&\boldsymbol{R}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))\end{aligned}$$

That $\dot{V}_2$ is bounded by Lemma 1.

$$\begin{aligned}&\dot{V}_2(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)\leqslant\\&-\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))[2\boldsymbol{P}_2\boldsymbol{A}_2(\boldsymbol{\alpha}^{(1)})^{-1}-\boldsymbol{P}_2\boldsymbol{W}_{21}\boldsymbol{Q}_2^{-1}\times\\&\boldsymbol{W}_{21}^{\mathrm{T}}\boldsymbol{P}_2-\boldsymbol{R}_1]\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t))-\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))\times\\&[2\boldsymbol{P}_3\boldsymbol{A}_3(\boldsymbol{\alpha}^{(2)})^{-1}-\boldsymbol{P}_3\boldsymbol{W}_{31}\boldsymbol{Q}_3^{-1}\boldsymbol{W}_{31}^{\mathrm{T}}\boldsymbol{P}_3-\boldsymbol{R}_2]\times\\&\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t))-\cdots-\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))[2\boldsymbol{P}_1\boldsymbol{A}_1(\boldsymbol{\alpha}^{(n)})^{-1}-\\&\boldsymbol{P}_1\boldsymbol{W}_{11}\boldsymbol{Q}_1^{-1}\boldsymbol{W}_{11}^{\mathrm{T}}\boldsymbol{P}_1-\boldsymbol{R}_n]\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t))-\\&\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t-\tau_1))[\boldsymbol{R}_1-\boldsymbol{W}_{12}^{\mathrm{T}}\boldsymbol{Q}_1\boldsymbol{W}_{12}]\times\\&\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))-\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t-\tau_2))\times\\&[\boldsymbol{R}_2-\boldsymbol{W}_{22}^{\mathrm{T}}\boldsymbol{Q}_2\boldsymbol{W}_{22}]\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))-\cdots-\\&\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t-\tau_n))[\boldsymbol{R}_n-\boldsymbol{W}_{n2}^{\mathrm{T}}\boldsymbol{Q}_n\boldsymbol{W}_{n2}]\times\\&\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))\end{aligned}$$

since $\boldsymbol{\Omega}_1>\boldsymbol{0}$, $\boldsymbol{\Omega}_2>\boldsymbol{0}$, $\cdots$, $\boldsymbol{\Omega}_n>\boldsymbol{0}$, there exists $\varepsilon_2>0$ such that $\boldsymbol{\Omega}_1-2\varepsilon_2\boldsymbol{I}_{l_2}>\boldsymbol{0}$, $\boldsymbol{\Omega}_2-2\varepsilon_2\boldsymbol{I}_{l_3}>\boldsymbol{0}$, $\cdots$, $\boldsymbol{\Omega}_n-2\varepsilon_2\boldsymbol{I}_{l_1}>\boldsymbol{0}$. Set $\boldsymbol{R}_1=\boldsymbol{W}_{12}^{\mathrm{T}}\boldsymbol{Q}_1\boldsymbol{W}_{12}+\varepsilon_2\boldsymbol{I}_{l_2}$, $\boldsymbol{R}_2=\boldsymbol{W}_{22}^{\mathrm{T}}\boldsymbol{Q}_2\boldsymbol{W}_{22}+\varepsilon_2\boldsymbol{I}_{l_3}$, $\cdots$, $\boldsymbol{R}_n=\boldsymbol{W}_{n2}^{\mathrm{T}}\boldsymbol{Q}_n\boldsymbol{W}_{n2}+\varepsilon_2\boldsymbol{I}_{l_1}$, which are positive definite and symmetric matrices. So

$$\begin{aligned}&\dot{V}_2(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)\leqslant\\&-\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))(\boldsymbol{\Omega}_1-2\varepsilon_2\boldsymbol{I}_{l_2}+\varepsilon_2\boldsymbol{I}_{l_2})\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t))-\\&\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))(\boldsymbol{\Omega}_2-2\varepsilon_2\boldsymbol{I}_{l_3}+\varepsilon_2\boldsymbol{I}_{l_3})\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t))-\cdots-\\&\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))(\boldsymbol{\Omega}_n-2\varepsilon_2\boldsymbol{I}_{l_1}+\varepsilon_2\boldsymbol{I}_{l_1})\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t))-\\&\varepsilon_2\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t-\tau_1))\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))-\\&\varepsilon_2\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t-\tau_2))\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))-\cdots-\\&\varepsilon_2\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t-\tau_n))\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))\leqslant\\&-\varepsilon_2\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t))\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t))-\\&\varepsilon_2\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t))\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t))-\cdots-\\&\varepsilon_2\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t))\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t))-\\&\varepsilon_2\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(t-\tau_1))\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(t-\tau_1))-\\&\varepsilon_2\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(t-\tau_2))\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(t-\tau_2))-\cdots-\\&\varepsilon_2\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(t-\tau_n))\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(t-\tau_n))\end{aligned}$$

Choose $\varepsilon_1>0$ such that $M\varepsilon_1\leqslant\varepsilon_2$, we have $\dot{V}(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)\leqslant-\varepsilon_1\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{A}_1\boldsymbol{v}_1(t)-\varepsilon_1\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{A}_2\boldsymbol{v}_2(t)-\cdots-\varepsilon_1\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{A}_n\boldsymbol{v}_n(t)$. Let $a=\min(a_{1,1},\cdots,a_{l_1,1},a_{1,2},\cdots,a_{l_2,2},\cdots,a_{1,n},\cdots,a_{l_n,n})$, $\theta=\max(\alpha_1^{(1)},\cdots,\alpha_{l_2}^{(1)},\alpha_1^{(2)},\cdots,\alpha_{l_3}^{(2)},\cdots,\alpha_1^{(n)},\cdots,\alpha_{l_1}^{(n)})$ and $r=\max(\|\boldsymbol{R}_1\|_2,\|\boldsymbol{R}_2\|_2,\cdots,\|\boldsymbol{R}_n\|_2)$. Obviously, $V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)$ is a positive definite and radically unbounded Lyapunov function. Take $\varepsilon>0$, such that

$$\varepsilon\varepsilon_1+\varepsilon p\theta-\varepsilon_1 a+r\theta^2\varepsilon\tau\,\mathrm{e}^{\varepsilon\tau}<0 \tag{11}$$

We obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\varepsilon t}V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t))=\\&\varepsilon\mathrm{e}^{\varepsilon t}V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)+\\&\mathrm{e}^{\varepsilon t}\frac{\mathrm{d}}{\mathrm{d}t}V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)\leqslant\\&\varepsilon\mathrm{e}^{\varepsilon t}[\varepsilon_1(\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{v}_1(t)+\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{v}_2(t)+\cdots+\\&\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{v}_n(t))+2\sum_{i_2=1}^{l_2}p_{i_2}^{(2)}\int_0^{v_{i_2,2}(t)}f_{i_2}^{(1)}(s)\mathrm{d}s+\\&\int_{t-\tau_1}^{t}\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(\tau))\boldsymbol{R}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(\tau))\mathrm{d}\tau+\\&2\sum_{i_3=1}^{l_3}p_{i_3}^{(3)}\int_0^{v_{i_3,3}(t)}f_{i_3}^{(2)}(s)\mathrm{d}s+\\&\int_{t-\tau_2}^{t}\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(\tau))\boldsymbol{R}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(\tau))\mathrm{d}\tau+\cdots+\\&2\sum_{i_1=1}^{l_1}p_{i_1}^{(1)}\int_0^{v_{i_1,1}(t)}f_{i_1}^{(n)}(s)\mathrm{d}s+\\&\int_{t-\tau_n}^{t}\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(\tau))\boldsymbol{R}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(\tau))\mathrm{d}\tau]-\\&\varepsilon_1\mathrm{e}^{\varepsilon t}(\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{A}_1\boldsymbol{v}_1(t)+\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{A}_2\boldsymbol{v}_2(t)+\cdots+\\&\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{A}_n\boldsymbol{v}_n(t))\end{aligned}$$

Noting that

$$2p_{i_2}^{(2)}\int_0^{v_{i_2,2}(t)}f_{i_2}^{(1)}(s)\mathrm{d}s\leqslant 2p\int_0^{v_{i_2,2}(t)}\alpha_{i_2}^{(1)}s\mathrm{d}s\leqslant p\theta v_{i_2}^2(t)$$

$$2p_{i_3}^{(3)}\int_0^{v_{i_3,3}(t)}f_{i_3}^{(2)}(s)\mathrm{d}s\leqslant 2p\int_0^{v_{i_3,3}(t)}\alpha_{i_3}^{(2)}s\mathrm{d}s\leqslant p\theta v_{i_3}^2(t)$$

$$\cdots$$

$$2p_{i_1}^{(1)}\int_0^{v_{i_1,1}(t)} f_{i_1}^{(n)}(s)\mathrm{d}s \leqslant 2p\int_0^{v_{i_1,1}(t)} \alpha_{i_1}^{(n)} s\mathrm{d}s \leqslant p\theta v_{i_1}^2(t)$$

where $p=\max(p_1^{(1)},\cdots,p_{l_1}^{(1)},p_1^{(2)},\cdots,p_{l_2}^{(2)},\cdots,p_1^{(n)},\cdots,p_{l_n}^{(n)})$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\varepsilon t}V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)) \leqslant \\
&\mathrm{e}^{\varepsilon t}(\varepsilon\varepsilon_1+\varepsilon p\theta-\varepsilon_1 a)(\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{v}_1(t)+ \\
&\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{v}_2(t)+\cdots+\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{v}_n(t))+ \\
&\varepsilon\mathrm{e}^{\varepsilon t}\Big[\int_{t-\tau_1}^{t}\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(v))\boldsymbol{R}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(v))\mathrm{d}v+ \\
&\int_{t-\tau_2}^{t}\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(v))\boldsymbol{R}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(v))\mathrm{d}v+\cdots+ \\
&\int_{t-\tau_n}^{t}\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(v))\boldsymbol{R}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(v))\mathrm{d}v\Big]
\end{aligned} \tag{12}$$

Integrating both sides of Eq. (12) from 0 to s, concerned with Eq. (11), similar to Theorem 2.3 in Lit. [6], we obtain

$$\begin{aligned}
&\mathrm{e}^{\varepsilon s}V(\boldsymbol{v}_1(s),\boldsymbol{v}_2(s),\cdots,\boldsymbol{v}_n(s),s)-V(\boldsymbol{v}_1(0),\boldsymbol{v}_2(0),\cdots,\boldsymbol{v}_n(0),0)\leqslant \\
&\int_0^s \mathrm{e}^{\varepsilon t}(\varepsilon\varepsilon_1+\varepsilon p\theta-\varepsilon_1 a)(\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{v}_1(t)+\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{v}_2(t)+ \\
&\cdots+\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{v}_n(t))\mathrm{d}t+r\theta^2\varepsilon\tau\mathrm{e}^{\varepsilon\tau}\int_{-\tau}^{0}\mathrm{e}^{\varepsilon v}(\boldsymbol{v}_1^{\mathrm{T}}(v)\boldsymbol{v}_1(v)+ \\
&\boldsymbol{v}_2^{\mathrm{T}}(v)\boldsymbol{v}_2(v)+\cdots+\boldsymbol{v}_n^{\mathrm{T}}(v)\boldsymbol{v}_n(v))\mathrm{d}v+ \\
&r\theta^2\varepsilon\tau\mathrm{e}^{\varepsilon\tau}\int_0^s \mathrm{e}^{\varepsilon v}(\boldsymbol{v}_1^{\mathrm{T}}(v)\boldsymbol{v}_1(v)+\boldsymbol{v}_2^{\mathrm{T}}(v)\boldsymbol{v}_2(v)+\cdots+ \\
&\boldsymbol{v}_n^{\mathrm{T}}(v)\boldsymbol{v}_n(v))\mathrm{d}v\leqslant \\
&r\theta^2\varepsilon\tau\mathrm{e}^{\varepsilon\tau}\int_{-\tau}^{0}\mathrm{e}^{\varepsilon v}(\boldsymbol{v}_1^{\mathrm{T}}(v)\boldsymbol{v}_1(v)+\boldsymbol{v}_2^{\mathrm{T}}(v)\boldsymbol{v}_2(v)+\cdots+ \\
&\boldsymbol{v}_n^{\mathrm{T}}(v)\boldsymbol{v}_n(v))\mathrm{d}v\equiv M_1\|\boldsymbol{\phi}\|_2^2
\end{aligned}$$

Therefore

$$\begin{aligned}
&V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)\leqslant \\
&(V(\boldsymbol{v}_1(0),\boldsymbol{v}_2(0),\cdots,\boldsymbol{v}_n(0),0)+M_1\|\boldsymbol{\phi}\|_2^2)\mathrm{e}^{-\varepsilon t}; \\
&\forall t>0
\end{aligned} \tag{13}$$

$$\begin{aligned}
&V(\boldsymbol{v}_1(0),\boldsymbol{v}_2(0),\cdots,\boldsymbol{v}_n(0),0)= \\
&\varepsilon_1(\boldsymbol{v}_1^{\mathrm{T}}(0)\boldsymbol{v}_1(0)+\boldsymbol{v}_2^{\mathrm{T}}(0)\boldsymbol{v}_2(0)+\cdots+ \\
&\boldsymbol{v}_n^{\mathrm{T}}(0)\boldsymbol{v}_n(0))+2\sum_{i_2=1}^{l_2}p_{i_2}^{(2)}\int_0^{v_{i_2,2}(0)}f_{i_2}^{(1)}(s)\mathrm{d}s+ \\
&2\sum_{i_3=1}^{l_3}p_{i_3}^{(3)}\int_0^{v_{i_3,3}(0)}f_{i_3}^{(2)}(s)\mathrm{d}s+\cdots+ \\
&2\sum_{i_1=1}^{l_1}p_{i_1}^{(1)}\int_0^{v_{i_1,1}(0)}f_{i_1}^{(n)}(s)\mathrm{d}s+ \\
&\int_{-\tau_1}^{0}\boldsymbol{f}^{(1)\mathrm{T}}(\boldsymbol{v}_2(v))\boldsymbol{R}_1\boldsymbol{f}^{(1)}(\boldsymbol{v}_2(v))\mathrm{d}v+ \\
&\int_{-\tau_2}^{0}\boldsymbol{f}^{(2)\mathrm{T}}(\boldsymbol{v}_3(v))\boldsymbol{R}_2\boldsymbol{f}^{(2)}(\boldsymbol{v}_3(v))\mathrm{d}v+\cdots+ \\
&\int_{-\tau_n}^{0}\boldsymbol{f}^{(n)\mathrm{T}}(\boldsymbol{v}_1(v))\boldsymbol{R}_n\boldsymbol{f}^{(n)}(\boldsymbol{v}_1(v))\mathrm{d}v\leqslant \\
&(\varepsilon_1+p\theta+r\theta^2\tau)\|\boldsymbol{\phi}\|_2^2\equiv M_2\|\boldsymbol{\phi}\|_2^2
\end{aligned}$$

According to Eq. (13) and the above inequality

$$\begin{aligned}
&\varepsilon_1\|\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t)\|_2^2= \\
&\varepsilon_1(\boldsymbol{v}_1^{\mathrm{T}}(t)\boldsymbol{v}_1(t)+\boldsymbol{v}_2^{\mathrm{T}}(t)\boldsymbol{v}_2(t)+\cdots+\boldsymbol{v}_n^{\mathrm{T}}(t)\boldsymbol{v}_n(t))\leqslant \\
&V(\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t),t)\leqslant \\
&(M_1+M_2)\|\boldsymbol{\phi}\|_2^2\mathrm{e}^{-\varepsilon t}
\end{aligned}$$

that is,

$$\|\boldsymbol{v}_1(t),\boldsymbol{v}_2(t),\cdots,\boldsymbol{v}_n(t)\|\leqslant\sqrt{\frac{M_1+M_2}{\varepsilon_1}}\;\|\boldsymbol{\phi}\|_2\mathrm{e}^{-\varepsilon t/2};\quad \forall t>0 \tag{14}$$

Inequality (14) implies that the origin of system (9) is global exponential stable.

4 Comparison with previous results

Now we compare our results with the previous result in Lit. [6], where authors gave a new sufficient condition for the existence, uniqueness and global stability of the equilibrium point for BAM neural network with constant delays:

$$\begin{aligned}
&\dot{u}_i(t)=-a_i u_i(t)+\sum_{j=1}^{m}w_{ij}s_j^{(1)}(z_j(t-\tau_1))+J_i^{(1)}; \\
&\qquad i=1,2,\cdots,n \\
&\dot{z}_j(t)=-b_j z_j(t)+\sum_{i=1}^{n}v_{ji}s_i^{(2)}(u_i(t-\tau_2))+J_j^{(2)}; \\
&\qquad j=1,2,\cdots,m
\end{aligned} \tag{15}$$

We could obtain the result in Lit. [6] from our work, when $n=2$, network (1) is similar to Eq. (15), Theorems (1), (2) became Lemma (2.2), Theorem (2.3) in Lit. [6].

Example 1 Assume the parameters in Eq. (9) are given as follows:

$$\boldsymbol{W}_1=\boldsymbol{W}_2=\cdots=\boldsymbol{W}_n=\boldsymbol{W}_{12}=\boldsymbol{W}_{22}=\cdots=\boldsymbol{W}_{n2}=\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&\ddots&1\\0&0&0&1\end{pmatrix}$$

and $\boldsymbol{A}_1=\boldsymbol{A}_2=\cdots=\boldsymbol{A}_n=a\boldsymbol{I}_n$, $\boldsymbol{Q}_1=\boldsymbol{Q}_2=\cdots=\boldsymbol{Q}_n=r\boldsymbol{I}_n$, $(\boldsymbol{\alpha}^{(1)})^{-1}=(\boldsymbol{\alpha}^{(2)})^{-1}=\cdots=(\boldsymbol{\alpha}^{(n)})^{-1}=\boldsymbol{P}_1=\boldsymbol{P}_2=\cdots=\boldsymbol{P}_n=\boldsymbol{W}_{11}=\boldsymbol{W}_{21}=\cdots=\boldsymbol{W}_{n1}=\boldsymbol{I}_n$, where $\boldsymbol{I}_n$ is

$n\times n$ identity matrix. Hence, we have

$$\boldsymbol{\Omega}_1=\boldsymbol{\Omega}_2=\cdots=\boldsymbol{\Omega}_n=2\boldsymbol{P}_2\boldsymbol{A}_2(\boldsymbol{\alpha}^{(1)})^{-1}-\boldsymbol{P}_2\boldsymbol{W}_{21}\boldsymbol{Q}_2^{-1}\boldsymbol{W}_{21}^{\mathrm{T}}\boldsymbol{P}_2-\boldsymbol{W}_{12}^{\mathrm{T}}\boldsymbol{Q}_1\boldsymbol{W}_{12}=\begin{pmatrix}2a-1/r & 0 & 0 & 0\\ 0 & 2a-1/r & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & 2a-1/r-rn\end{pmatrix}$$

If $a>\sqrt{n}$ and is sufficiently large, there exists positive constant r such that $2a-1/r-rn>0$ and $2a-1/r>0$. Hence the condition (T) of Theorems 1 and 2 is satisfied, the origin of system (9) is globally exponential stable.

5 Conclusion

We study a class of neural networks with constant delays in this paper, comparing with previous work[6], we expand the result of neural network from 2-layer to n-layer by constructing Lyapunov function. Our result includes the result of work in Lit. [6].

References:

[1] HAN Wei, LIU Yan, WANG Linshan. Robust exponential stability of Markovian jumping neural networks with mode-dependent delay [J]. **Communications in Nonlinear Science and Numerical Simulation**, 2010, **15**(9):2529-2535.

[2] WANG Yangfan, LU Chunge, JI Guangrong, *et al*. Global exponential stability of high-order Hopfield-type neural networks with S-type distributed time delays [J]. **Communications in Nonlinear Science and Numerical Simulation**, 2011, **16**:3319-3325.

[3] WANG M, WANG L. Global asymptotic robust stability of static neural network models with S-type distributed delays [J]. **Mathematical and Computer Modelling**, 2006, **44**:218-222.

[4] YANG Fengjian, ZHANG Chaolong, CHEN Chuanyong, *et al*. Global exponential stability of a class of neural networks with delays [J]. **Acta Mathematicae Applicatae Sinica**, 2009, **25**(1):43-50.

[5] ZHAO Weirui, ZHANG Huanshui. Globally exponential stability of neural network with constant and variable delays [J]. **Physics Letters A**, 2006, **352**(4/5):350-357.

[6] ZHAO Weirui, ZHANG Huanshui, KONG Shulan. An analysis of global exponential stability of bidirectional associative memory neural networks with constant time delays [J]. **Neurocomputing**, 2007, **70**(7/9):1382-1389.

[7] DING Ke, HUANG Nanjing, XU Xing. Global robust exponential stability of interval BAM neural network with mixed delays under uncertainty [J]. **Neural Processing Letters**, 2007, **25**(2):127-141.

[8] LI Chuandong, LIAO Xiaofang, ZHANG Rong. Delay-dependent exponential stability analysis of bi-directional associative memory neural networks with time delay: an LMI approach [J]. **Chaos, Solitions & Fractals**, 2005, **24**(4):1119-1134.

[9] FORTI M, TESI A. New conditions for global stability of neural networks with application to linear and quadratic programming problems [J]. **IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications**, 1995, **42**(7):354-366.

带有常时滞循环耦合神经网络的全局指数稳定性

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摘要： 讨论了带有常时滞循环耦合神经网络的全局指数稳定性，在讨论过程中通过构造同胚映射论证了该系统平衡点的存在性与唯一性，再通过构造合适的 Lyapunov 函数论证唯一平衡点是全局指数稳定的. 类似于已有的神经网络稳定性方面工作，在神经元的激励函数满足 Lipschitz 条件且相关系数构成矩阵也满足给定条件下，得到 n 层带有常时滞的神经网络全局指数稳定的动力学性质. 所得结果同时也蕴含当神经元的衰减速率足够大时，神经网络是全局指数稳定的.

关键词： 指数稳定性；平衡点；神经网络；Lyapunov 函数

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